# A PRIMER

OF

# CALCULUS

BY

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New York

THE MACMILLAN COMPANY

LONDON: MACMILLAN & CO., LTD.

1901

MOORE & LANGEN PRINTING CO.,
TERRE HAUTE, IND.

# PREFACE.

This Primer has been written to meet the needs of the author, first, for a primary course in the calculus, and secondly, for an outline of topics in a more advanced course that is suitable for combined lecture and text book instruction.

The author's method of development is essentially Newton's method of fluxions, as presented by Hamilton in his Elements of Quaternions, Bk. III, ch. II. This method is clear, logical, and scientific, and it deserves more recognition than it has received in general analysis, if for no other reason than that it is the method of the original discoverer of the calculus. Its failure to be adopted is due to want of early publication and defective notation, since it is remarkably perfect and general in principle. The subsequent discoverer, Leibintz, gained the field by publications in a desirable notation, although founded upon inferior infinitessimal principles. Lagrange attempted a modification of the infinitessimal into the idea of a principal part as determined by first terms of expansions, and made the "differential co-efficient" the primary quantity. Modern text books have returned to Newton's method of limits as applied to Lagrange's differential co-efficient; there is here offered a complete return to Newton, with the fluxion or differential as the primary quantity.

The point of view of our development is that differentiation is an arithmetical process and that its resulting differentials are numbers like other numbers, which are classified as independent or dependent variables according to the like character of the variables from which they are derived. The usefulness of a process consists in its practical applications, but nothing is gained by attempting to show practical utility before the fundamental principles and rules of differentiation are fairly mastered. It must be accepted at first that differentiation is of as much higher order of practical value than the usual processes of arithmetic as it is in advance of those processes in respect to fundamental ideas. Also, the student will have more confidence in the use of the calculus when he learns it first as a rigorous and exact arithmetical method. The following course of lessons brings the student as rapidly as it is desirable into practical applications.

One object of the author has been to discourage empirical acquisition through illustrated examples worked out in full. If a student is not able to follow out careful instructions as to how to do his work, without having it done for him, he is lacking in the first elements of an engineer at least, and it is time that he began mental training in that direction (even if it is a little hard.)

Attention is called to the note on page 6 which explains a general principle of notation based on $dx^2 = dx \,.\, dx$. While unusual in the case of trigonometric functions, yet it is clear; *i.e.*, there is no conflict between $\sin x^2 = \sin x \,.\, \sin x$ and $\sin \,.\, x^2 = \sin (x^2)$. It also removes several inconsistencies in trigonometric notation that many students do not understand. The ordinary notation may be used, however, if desired.

ARTHUR S. HATHAWAY.

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# A PRIMER OF CALCULUS

## CHAPTER I

### Differentiation

**1.** VARIABLE AND CONSTANT QUANTITIES. Letters that do not denote special numbers, as do the letters $\pi=3.14159...$ and $e=2.71828...$, but which stand for undetermined numerical values in a given problem, are called variables or constants, according as their values are considered to change in that problem or not. Unless the contrary is stated, first letters of the alphabet will denote constants, as $a$, $b$, $c$, . . . , and final letters of the alphabet will denote variables, as $u$, $v$, $w$, $x$, $y$, $z$.

**2.** INDEPENDENT AND DEPENDENT VARIABLES. The independent variables are those whose values are assigned at will, each without reference to the value of any other variable. The dependent variables are those whose values depend upon and are determined by the values of one or more of the independent variables. Thus, $x, y$ being independent variables, then $u=x^2, v=y^2, w=xy$ are dependent variables. Every problem in which variation is possible has a certain number of independent variables; the remaining variables are dependent, and in a general sense, each is expressible in terms of the independent variables and constants, so that always as many equations connect all the variables as there are dependent variables.

**3.** FUNCTIONS. A variable whose value depends upon and is determined without ambiguity by the values of certain other variables is called *a function of those variables.* Thus $x^2$ is a function of $x$, and $y^2$ is a function of $y$, and $xy$ is a function of $x$ and $y$. In general, any expression that involves several variables, whose value is computable by means of the values of its component variables, is a function of those variables. Conversely, any function of given variables can have, for its representation, an expression involving the variables upon which it depends. Suppose, for example, a function were known for which no expression in terms of its variables existed; then it would be proper to make an expression that should always stand for the value of the function corresponding to the values of the variables given in the expression. This was the case, for example, with the logarithmic and trigonometric functions when they were first considered, and the special symbols log, sin, cos, tan, sec, etc., have been introduced as characteristic symbols for these functions, so that $\sin x$ denotes the value of the function whose characteristic symbol is sin, corresponding to any value of $x$.

**4.** We shall often use letters as characteristic symbols of *undetermined* functions, and not as undetermined numbers, particularly the letters $f$, $F$, because each is the first letter in the word "function." Thus $fx$ will denote an undetermined function of $x$ that can be chosen as we please, and $f(x, y)$ and undetermined function of $x$ and $y$. To make $fx$ one function or another is to identify $f$ with the process or characteristic of the function. Thus $fx = \sin x$ makes $f = \sin$; $fx = x^2$ makes $f =$ "square of"; $fx = x^2 - 2x + 3$, makes $f =$ "square of, minus the double of, plus 3"; $f(x, y) = 3x^2 + 2xy - y^2$ makes f stand for computing three times the square of the first variable named, plus twice the product of the first and second, minus the square of the second.

**5.** ORIGINAL VALUES. The original values of the variables are those which we suppose designated by the symbols of the variables, so that they are either undetermined values, or else the numerical values that may be assigned to such symbols; thus $x$, $y$, $x^2$, $y^2$, $xy$, are undetermined original values, and $x=3$, $y=4$, $x^2=9$, $y^2=16$, $xy=12$ are assigned original values. The advantage of leaving the original values of the variables undetermined or literal, when carrying out processes of computation with them, is that one such literal development involves the results of all possible assumptions of numerical value.

**6.** NEW VALUES. New values of the variables will be new symbols of the variables (in general the old symbols accented) that stand for undetermined or determined values that are in general different from the original values. Thus, if $x$, $y$ be original values of two independent variables, then $x'$, $y'$ would denote new values of those variables, and $x^2$, $y^2$, $xy$ are the original values, and $x'^2$, $y'^2$, $x'y'$ are the new values of the squares and the product of the independent variables. As numerical cases, taking $x=3$, $y=4$, we could then take $x'=4$, $y'=7$, so that 3, 4, 9, 16, 12 are original values of the independent variables and their squares and product, and 4, 7, 16, 49, 28 are new values of the same. Again instead of these new values $x'=4$, $y'=7$, take new values $x'=3.1$, $y'=4.3$, that are nearer the original values $x=3$ $y=4$, and the corresponding new values of their squares and product are 9.61, 18.49, 13.33, which are also nearer the original values than at first. When we speak of the values of the variables, without qualifying them as new values, we always mean the original values. The original values of the variables denote, in other words, *the* values of the variables that are being considered. The new values are temporary values that are to be considered as approaching the original values; and they are introduced, and made nearer and nearer the original values, for the purpose of determining some questions of variation of the variables *at* their original values, just as, in order to determine the motion of a train *at* a given instant, it is practically necessary to consider its motion for a very small time thereafter, with the knowledge that greater and greater accuracy is attained the smaller this time is taken, so long as it can be accurately measured together with the corresponding distance passed over by the train.

**7.** Differences. The changes of value of variable quantities from original to new values are called *differences of the variables.* A difference is denoted by prefixing the Greek letter delta ($\Delta$) to the symbol or literal value of the variable. Thus

$$\Delta x = \text{"the difference of } x\text{"} = x' - x,$$

$$\Delta y = \text{"the difference of } y\text{"} = y' - y.$$

$$\Delta(x^2) = \text{"the difference of } x^2\text{"} = x'^2 - x^2.$$

$$\Delta(y^2) = \text{"the difference of } y^2\text{"} = y'^2 - y^2.$$

$$\Delta(xy) = \text{"the difference of } xy\text{"} = x'y' - xy.$$

In the numerical cases

$$x=3,\ y=4, \text{ and } x'=4,\ y'=7,$$

we have,

$$\Delta x=1,\ \Delta y=3,\ \Delta(x^2)=7,\ \Delta(y^2)=33,\ \Delta(x\,y)=16;$$

and for the new values $x'=3.1$, $y'=4.3$, that are nearer the original values, we have the smaller differences,

$$\Delta x=.1,\ \Delta y=.3,\ \Delta(x^2)=.61, \Delta(y^2)=2.49,\ \Delta(xy)=1.33.$$

**8.** *The difference of an Independent Variable is a New Independent Variable.* In other words, if $x$ be an independent variable, then $\Delta x=x'-x$, may be any change of value we please. In fact the new value, $x'$, depends upon the original value $x$, and the change of value $\Delta x$, viz., $x'=x+\Delta x=$ original value plus the difference or change of value. The value of $\Delta x$ depends neither upon the value of $x$ nor upon that of any other variable, but can be taken whatever value we please. If, however, $x$ be what is called a *real* independent variable, i. e., one limited to real values only, then $\Delta x$ must also be a real independent variable. In fact, generally, the limitation of all values of a variable to real values also limits its changes of value or differences to real values.

**9.** *The difference of a dependent variable is a new dependent variable, whose independent variables are the original independent variables and their differences.* Thus take the square of an independent variable, as $x^2$, then when $x$ and $\Delta x$ are assigned we have

$$x'=x+\Delta x, \text{ and } x'^2=x^2+2x\Delta x+\Delta x^2,*$$

$$\text{so that } \Delta(x^2)=x'^2-x^2=2\,x\,\Delta\,x+\Delta x^2,$$

which depends upon $x$, $\Delta\,x$, as required.

Similarly $\Delta(xy)=x'y'-xy=(x+\Delta x)(y+\Delta y)-xy$ $=x\Delta y+y\Delta x+\Delta x\Delta y$, which depends upon $x$, $y$, $\Delta x$, $\Delta y$, as required. In general, if $w=f(x, y)$, then $w'=f(x', y')$ $=f(x+\Delta x, y+\Delta y)$, so that

$$\Delta w=f(x+\Delta x, \ y+\Delta y)-f(x, y),$$

which depends upon $x$, $y$, $\Delta x$, $\Delta y$. It appears also from this result, that even when the variables of a function are not independent variables, the difference of such function will depend upon its variables and their differences in exactly the same way as if the variables were independent.

**10.** PROPORTIONAL DIFFERENCES. Equimultiples of simultaneous differences by the same *real* proportional factor will be called *proportional differences;* they are

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*Such symbols as $\Delta x, f\,x$, $\sin x$, $\log x$, etc., which are not separable into *number* factors, because one of the factor symbols is a characteristic, and not a number, are equivalent to single symbols of number, and exponents to such a symbol should be regarded as applying to the symbol as a whole, when no parenthesis or dot intervenes to make a separation of its parts. Thus $\Delta x^2=\Delta x\Delta x$, and not *the difference of* $x^2$. The latter difference is written $\Delta(x^2)$ or $\Delta.x^2$. A similar symbol is $\Delta^2x=\Delta\,\Delta x=$ difference of $\Delta x$, regarded as a new independent or dependent variable according as $x$ is independent or dependent. However, by force of usage, and contrary to principles of notation, $\sin^2 x$ means $\sin x^2$ and not sin ($\sin x$), and similarly for the squares or other powers of all trigonometric functions, except for the exponent $-1$. Thus $\sin^{-1}x$ is not $\sin x^{-1}$, but conforms again to general principals of notation in which $f^{-1}x$ stands for "that function whose f is $x$," so that always $ff^{-1}x=x$.

any proportionals to the differences, if we understand the term proportion in the sense that the ratio is a real and not an imaginary number. The literal symbol of the proportional factor will be $N$, so that $N\Delta x$, $N\Delta y$, $N\Delta(x^2)$, $N\Delta(y^2)$, $N\Delta(xy)$ denote proportional differences of the variables $x$, $y$, $x^2$, $y^2$, $xy$.

*E. g.*, let $x=3$, $y=4$, $\Delta x=1$, $\Delta y=3$, $N=4$, then

$$\begin{array}{lllll} N\Delta x, & N\Delta y, & N\Delta(x^2), & N\Delta(y^2), & N\Delta(xy) \\ =4, & 12, & 28, & 132, & 64. \end{array}$$

For the new values $x'$, $y'$, $=3.1$, $4.3$, corresponding to $\Delta x=.1$ $\Delta y=.3$, which are nearer the original values than before, and the larger factor $N=49$, we find,

$$\begin{array}{lllll} N\Delta x, & N\Delta y, & N\Delta(x^2), & N\Delta(y^2), & N\Delta(xy) \\ =4.9, & 14.7, & 29.89, & 122.01, & 65.17. \end{array}$$

Note that although the differences have all been decreased in value from their first values, yet the corresponding increase in the proportional factor has left the proportional differences about the same as before.

**11.** DIFFERENTIATION. Differentiation is the process of finding limits of proportional differences of variable quantities, as the differences tend toward zero and the proportional factor tends towards infinity. Such limits are called *differentials of the variables.* A differential is denoted by prefixing the letter $d$ as characteristic of differentiation to the literal value of the variable. Thus, for independent variables, $x$, $y$,

$$dx = \textit{differential of } x = \lim N\Delta x = \lim N(x'-x),$$
$$dy = \textit{differential of } y = \lim N\Delta y = \lim N(y'-y).$$

The hypotheses of this differentiation of independent variables, $x$, $y$, are firstly, that the differences approach zero (or the new values approach the old) while the proportional factor correspondingly approaches infinity, so that the proportional differences approach limits, and secondly, that these limits are designated by $dx$, $dy$.

Turning to dependent variables, we have similarly

$$d(x^2)=\lim\ N\Delta(x^2)=\lim\ N(x'^2-x^2)$$
$$=\lim.[(x'+x)\ N(x'-x)]=2xdx\ ;*$$

$$d(y^2)=\lim\ N\Delta(y^2)=\lim\ N(y'^2-y^2)$$
$$=\lim\ [(y'+y)N(y'-y)]=2y\,dy;$$

$$d(xy)=\lim\ N\,\Delta(xy)=\lim\ N(x'y'-xy)$$
$$=\lim\ [y'N(x'-x)+xN(y'-y)]=ydx+xdy.$$

As further exercises show similarly that $d(x^3)=3x^2dx$, $d\frac{1}{x}=-\frac{dx}{x^2}$, $d\sqrt{x}=\frac{dx}{2\sqrt{x}}$, $d(x^2y^3)=2xy^3dx+3x^2y^2dy$.

**12.** To understand differentiation, and the exact signification of the resulting differentials as variable numbers, some points in the process of differentiation must be discussed more fully, and in particular they must be illustrated by numerical values.

**13.** *The differentials of independent variables are new independent variables.* In illustration, to make $dx=5$, we may take successively

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*Observe that since $x'$ approaches $x$, therefore $x'+x$ approaches $2x$, and also that $N$ increases as $x'$ approaches $x$, so that $N(x'-x)$ approaches the limit, $dx$. The limit of a product being the product of the limits of its factors, we therefore find that $\lim\ [(x'+x).N(x'-x)]=2xdx$.

$\Delta x = 1, .1, .01, .001, \ldots .$ limit$=0$

$N = 4, 49, 499, 4999, \ldots .$ limit$=\infty$;

then $N\Delta x = 4, 4.9, 4.99; 4.999, \ldots .$ limit$=5$.

Thus, in this way, we determine

$$dx = \lim N\Delta x = 5.$$

If instead of the above series of values of $N$ we should take another in which every proportional factor becomes double its preceding value so that we have successively,

$$N = 8, 98, 998, 9998, \ldots .$$

then with the same series of values of $\Delta x$ as before wc should find,

$N\Delta x = 8, 9.8, 9.98, 9.998, \ldots .$ limit$=10$,

which gives another value $dx, = \lim N\Delta x, = 10$, which is also double the preceding value of $dx$. In general $dx$ can be determined any value we please without regard to the value of $x$ or of any other variable, since the value of $\Delta x$ may be assigned at will, and its series of values approaching zero, assigned likewise as we please, in respect to value and law of continuation, so that whatever series of values approaching infinity may have been already assigned to $N$, we can make the proportional difference $N\Delta x$ take a series of successive values that will approach any limit we please.

Take, for example, $\Delta x = \frac{a}{N}$, which gives, as $N$ increases indefinitely, a corresponding value of $\Delta x$ that is approaching zero as required; then the series of values of N $\Delta x$ will be

$$N\Delta x = a, \ a, \ a, \ldots .$$

whose limit approached is $a$. Or take

$\Delta x = a\,N/(N^2+5) = a/(\mathrm{N}+\frac{5}{N})$ which approaches zero; then

$$N\Delta x = aN^2/(N^2+5) = a/(1+\frac{5}{N^2}),$$

which approaches $a$.

There are, in fact, an innumerable number of different ways of making each independent difference approach zero, and the common proportional factor approach infinity, so that the proportionals of those differences shall each approach any assigned value we please. If, however, we are considering a *real* independent variable, $x$, then since $N$ and $\Delta x$ are real; therefore $N\Delta x$ is real and must approach a real value. In words, ***the differential of a real independent variable is a new real independent variable.***

**14.** Understand that in the differential process such as $\Delta x = 1, .1, 01, .001, \ldots,$ and $N = 4, 499, 49, 4999, \ldots,$ in which the limit of $N\Delta x$ is sought, we do not consider $\Delta x$ as ever actually zero, or $N$ as actually infinity, so that we are not trying to find a value of "infinity times zero." In fact, a little common sense will show that since neither zero nor infinity are any actual values, therefore "infinity times zero" is a phrase that is in itself meaningless. Nor can this phrase be given a definite meaning in accordance with the usual acceptance of zero as denoting the nominal limit of a value that becomes smaller and smaller *without limit of smallness*, and of infinity as the nominal (but not-existing) limit of a value that becomes larger and larger *without limit of largeness.* Since one factor of a product can become smaller and smaller, and the other factor larger and larger, so that the product shall approach any value we please, it follows that even these limit ideas of zero and infinity cannot give determinate significance to infinity times zero. Should the student see cause, from these facts, to object to the independent differentials as too indeterminate in value for mathematical consideration, then the same objection would be equally valid against any independent variables, and against the whole idea of variation of value, which must be founded on the initial idea of certain indeterminate values, which can receive or change value at will, and of other related values which depend upon these undetermined or independent values.

**15.** *The differential of a dependent variable is a new dependent variable that is dependent upon and determined by (i. e., a function of) the independent variables and their differentials.* This result is a matter of definition and as a test of differentiability. For the only way in which the limit of the dependent proportional difference might be changed in value, *without changing the values of the independent variables and their differentials,* would be to take different series of values of the independent proportional differences, but not so as to change their assigned limits; and when such variations of approach, alone cause variations in the limit of the dependent proportional difference, we may consider that there is no definite limit or differential, and that the dependent variable is therefore *non-differentiable.* The test of differentiability is therefore the determination of the dependent differential solely in terms of the independent variables and their differentials.

**16.** For example, $x^2$ is differentiable, because $d(x^2)=\lim N\Delta(x^2)=\lim (x'+x).\,N(x'-x)=2x\,dx$, an expression that is definitely obtained in whatever way we suppose $x'$ to take an indefinitely continued series of values that approach the limit $x$ at the same time that $N$ takes a corresponding series of larger and larger values so that $N(x'-x)$ approaches the limit $dx$. Similarly $x\,y$ is differentiable, because we find invariably

$$\lim N\Delta(x\,y)=\lim (y'N\Delta x+xN\Delta y)=y\,dx+x\,dy,$$

in whatever manner $\Delta x$, $\Delta y$ approach zero and $N$ approaches infinity so that we have $\lim N\Delta x=dx$, $\lim N\Delta y=dy$ It will be found that all "continuous" variables,* for which expressions known to the student exist, are differentiable, except that in some expressions, for certain values of the variables involved, it

*"Continuous" means varying by small amounts when the variables change by small amounts, the dependent change approaching zero when the independent changes do so. The expression $fx=x+$ integer part of $x$, is not continuous at integral values of $x$; viz. when $x'$ increases towards 2, $f\,x'$ increases toward 3, but $f\,2=4$, so that $f\,x'-f\,x$ does not tend to vanish as $x'$ approaches $x=2$.

may happen that the value of the differential is ambiguous. This will be shown in the differential expression itself, so that it need not be regarded as affecting the general differentiable character of the variable in question.

**17.** *The differential of a function of one or more variables is the same function of its variables and their differentials, whether the variables are all independent or one or more of them are dependent.*

This is a consequence of the definition of differentiability which makes a function, $w=f(x, y)$ that is differentiable have a differential,

$$dw=\lim N\Delta w=\lim N[f(x+\Delta x,\ y+\Delta y)-f(x, y)],$$

that is a definite expression in terms of $x$, $y$, and $dx=\lim N\Delta x$, $dy=\lim N\Delta y$; say $dw=f'(x, y, dx, dy)$. If such a result holds when $x$, $y$ are independent variables, so that we have arbitrary methods of making $\lim N\Delta x=dx$, $\lim N\Delta y=dy$, in which $dx$, $dy$, are arbitrarily selected values, then it must all the more be true when we have only certain dependent methods of making $\lim N\Delta x=dx$, $\lim N\Delta y=dy$, where $dx$, $dy$ are dependent values. In other words, the dependent methods of approach, and the dependent limits, are included among the arbitrary methods of approach and the arbitrary limits. Thus

$$d(y^2)=\lim (y'+y)N\Delta y=2y\,dy,$$

whether $y$ is independent, so that $\lim N\Delta y=dy$ is also independent, or whether $y$ is dependent, so that $\lim N\Delta=dy$ is also dependent. In the latter case there remains the finding of $dy$ in proper terms, from the value of $y$ in terms of the independent variables, before the differentiation of $y^2$ can be considered as completed. Similarly $d(xy)=ydx+xdy$, whether $x$, $y$ are independent or dependent variables; and if we have actually $y=x^2$, so that $dy=2xdx$, then $xy=x^3$ and $d(x^3)=x^2dx+x\,.2x\,dx=3\,x^2dx$. Although this result is obtained indirectly yet it must verify directly. Thus,

$$N\Delta(x^3)=N(x'^3-x^3)=(x'^2+x'x+x^2)N(x'-x)$$

whose limit [as $x'$ approaches $x$ and $N$ increases so that $N(x'-x)$ approaches $dx$] is easily seen to be $3x^2\mathrm{d}x$. Again, in $d(x\,y)=y\,dx+x\,dy$, we can put $y=x^3$ so that $d\,y=3x^2\,d\,x$, and making these substitutions for $y$ and $dy$, we find

$$\mathrm{d}(x^4)=x^3dx+x.3x^2\mathrm{d}x=4x^3\mathrm{d}x.$$

Let the student verify this result directly, and also go over the differentiation of the product $xy$ which gives the value $d(xy)=ydx+xdy$, and try to find how any supposed dependence of $y$ upon $x$ could do more than make $dy$ correspondingly dependent upon $x$ and $dx$ (assuming, of course, that the given dependence of $y$ upon $x$ makes it a differentiable function.)

**18.** The differential of a given function is therefore seen to be a fixed rule for differentiating that function, even when its variables, instead of being simple independent variables, are any complex functions of other variables. Thus from $d(u^2)=2\,udu$, we have it equally true, by replacing $u$ by $x^2+y^2$, that

$$d.(x^2+y^2)^2=2(x^2+y^2)d(x^2+y^2),$$

which finally reduces to

$$4(x^2+y^2)(x\,dx+y\,dy),$$

since we will find that

$$d(x^2+y^2)=2(x\,dx+y\,dy).$$

It is on this account that rules of differentiation become important and of wide application, whether expressed in terms of one set of letters or another, since it will be indifferent what letters are employed to denote the variables. In fact the more important rules are best memorized in words.

**19.** Thus,

$$d(xy)=y\,dx+x\,dy,$$

is in words: *The differential of a product of variables equals the sum of the products consisting severally of the differential of each factor into the remaining factor.* This rule extends, also, to a product of any number of factors, *e. g.*,

$$d(x\,y\,z)=y\,z\,dx+x\,z\,dy+x\,y\,dz;\text{ etc.}$$

To prove this, let the product $x\,y\,z$ change to $x'\,y\,z$, then to $x'\,y'\,z$, then to $x'\,y'\,z'$. This is a succession of partial changes of value due to first changing $x$ alone, then $y$ alone, then $z$ alone, and the sum of these partial changes equals the total change. Thus,

$$x'y'z'-x\,y\,z=(x'yz-x\,y\,z)+(x'y'z-x'yz)+(x'y'z'-x'y'z)$$

or

$$\Delta(x\,y\,z)=y\,z\Delta x+x'z\Delta y+x'y'\Delta z.$$

Multiplying this equation by $N$ and remembering that

$$d=\lim N\Delta,$$

and

$$\lim x'=x,\ \lim y'=y,\ \text{etc.},$$

we find,

$$d(x\,y\,z)=y\,z\,dx+x\,z\,dy+x\,y\,d\,z.$$

**20.** *When the values of the variables of a function are assigned, then the value the differential of the function varies proportionally with the values of the differentials of its variables.* For, let $x, y,\cdots$, be the assigned values of the variables, and $w$ the corresponding value of the function; then $\Delta w$ will be assigned when $\Delta x$, $\Delta y,\cdots$ are assigned, and lim $N\Delta w$ is determined when lim $N\Delta x$, lim $N\Delta y,\cdots$ are determined. If the latter limits be made $x_1$, $y_1,\cdots$ and the former consequently becomes $w_1$, then to make the latter change proportionally to $kx_1$, $ky_1,\cdots$, we have only to take new multipliers each k times as large as before, with the same values of $\Delta x$, $\Delta y\cdots$ as before, since

$$\lim kN\Delta x=k\lim N\Delta x=k\,x_1, \lim kN\Delta y=k\lim N\Delta y=ky_1,$$

etc. But in this method of approach each $\Delta w$ remains the same as before, and the limit of the new proportional difference is

$$\lim k\,N\Delta\,w=\text{k}\lim N\Delta\,w=k\,w_1.$$

In other words, if $w_1$, $x_1$, $y_1,\cdots$ be corresponding values of $dw$, $dx$, $dy,\cdots$, and we change $dx$, $dy,\cdots$ proportionally to new values $kx_1$, $ky_1,\cdots$ then $dw$ changes in the same proportion to the new value $kw_1$.

**21.** Since proportional factors must be real numbers, it follows that the proportional factor $k$ of the preceding proposition must be real and not imaginary. An important consequence of that proposition is that: *In the differential of a function of one real variable, the differential of the variable appears only as a factor of the result*, or,

$$d\,fx = f'x\,dx$$

where $f'x$ is a function of $x$ called *the differential coefficient* of $fx$ as to $x$, and also, *the derivitive* of $fx$ as to $x$. In fact $x$ being assigned, if any two values of $dx$ are in the ratio k : 1, (where $k$ must be real because $x$ and therefore $dx$ are real variables) then the corresponding values of $dfx$ are in the same ratio by Art. 20; thus the quotient $dfx/dx$ does not change value when $dx$ changes value; and thence this quotient depends on the value of $x$ alone, so that it is some function, $f'x$, of $x$.

**22.** The theorem of Art. 21 does not hold for all functions, when the variable is not limited to real values. Thus if $z = x + y\sqrt{-1}$ be an imaginary variable whose real components are $x$, $y$, then

$$\text{mod}\, z = \sqrt{(x^2 + y^2)}$$

is a function of $z$, whose differential will be, as the student may verify by the work in full,

$$d \,\text{mod}\, z = (x\,dx + y\,dy)/\text{mod}\, z.$$

If this differential contain $dz = dx + \sqrt{-1}.dy$ as a factor only, so as to be of the form $f'z\,dx + \sqrt{-1}f'z\,dy$ whatever values $dx$, $dy$ may have, then

$$f'z = x/\text{mod } z = y/(\text{mod } z\sqrt{-1}), \text{ or } x = -y\sqrt{-1}$$

which is impossible, remembering that $x$, $y$ are any real values. On the contrary

$$d.z^2 = 2zdz,\ d.z^3 = 3z^2dz, \text{ etc.}$$

**23.** ANALYTICAL FUNCTIONS. Any differentiable function of one variable, whose differential contains the differential of its variable only as a factor, is called an *analytical function.* Any function of a *real* variable is (art 21) an analytical function; but for an imaginary variable $z$, mod $z$ is not an analytical function of $z$, while $z^2$, $z^3$, etc., are such.

**24.** DERIVATION. Derivation is the process of differentiation followed by division by the differential of a variable. The result of derivation is *the derivative of the function as to the variable*, and must be a function of the variable alone if derivation is possible. In other words, derivation is a process that is applicable only to analytical functions of one variable. Derivation can have a definition of its own not depending upon differentiation, viz., it is the process of finding the limit of the quotient of the difference of the function by the difference of the variable as the differences approach zero, provided there is a definite limit depending on the value of the variable alone, and not at all upon the manner of approach of its difference to zero. This follows from

$$\frac{dfx}{dx} = \lim \frac{N\Delta fx}{N\Delta x} = \lim \frac{\Delta fx}{\Delta x}.$$

EXAMPLES. I.

1. If $x=3$, $y=4$, and we take successively $\Delta x=1$, .7, .07, .007, and so on smaller and smaller, $\Delta y=2$, 1.3, .13, .013, and so on smaller and smaller, then find the corresponding series of values of

$$\Delta(x^2),\ \Delta(y^2),\ \Delta(xy).$$

2. If in Ex. 1, we also take successively

$$N=1,\ 9,\ 99,\ 999, \text{ and so on, larger and larger}$$

show that we thus determine

$$dx=7,\ dy=13,\ d(x^2)=42,\ d(y^2)=104,\ d(xy)=67,$$

and verify the last three from their literal values

$$d(x^2)=2xdx,\ d(y^2)=2ydy,\ d(xy)=ydx+xdy.$$

3. If in Ex. 2 we double each value of $N$ in its series of values, show by full numerical computation, that the values of $dx$, $dy$ and also those of the dependent differentials are doubled.

4. Show that, if $x=3$, $y=4$, then however we make $\Delta x$, $\Delta y$ approach zero and $N$ approach infinity so that $N\Delta x$, $N\Delta y$, approach 7, 13, respectively, we shall have $N\Delta(x^2)$, $N\Delta(y^2)$, $N\Delta(xy)$ approaching the limits 42, 104, 67, respectively.

$$[N\Delta(x^2)=N[(3+\Delta x)^2-9]=6.N\Delta x+\Delta x.N\Delta x, \text{ etc.}]$$

5. Prove that $d(z^2)=2zdz$, $d(y^3)=3y^2dy$, $d(y^6)=6y^5dy$. Also prove the last equation from the preceding ones, by putting $z=y^3$.

6. Prove that

$$d\frac{1}{x}=-\frac{dx}{x^2},\quad d\frac{1}{y^2}=-\frac{2dy}{y^3},\quad d\frac{1}{x^4}=-\frac{4dx}{x^5}.$$

Also verify the last equation by taking $y=x^2$ in the second.

7. Prove that

$$d\sqrt{z}=dz/2\sqrt{z},\quad d\sqrt{(x^2+y^2)}=(xdx+ydy)/\sqrt{(x^2+y^2)}$$

Also verify the last equation by putting $z=x^2+y^2$ in the first. Also verify the first from $(\sqrt{z})^2=z$.

8. Prove that $d\sqrt{(a^2+s^2)}=sds/\sqrt{(a^2+s^2)}$.

9. Prove that $d.y^{\frac{4}{3}}=\frac{4}{3}y^{\frac{1}{3}}dy$.

[$\Delta.y^{\frac{4}{3}}=y'^{\frac{4}{3}}-y^{\frac{4}{3}}=(y'^4-y^4)/(y'^{\frac{8}{3}}+y'^{\frac{4}{3}}y^{\frac{4}{3}}+y^{\frac{8}{3}})$, etc., or let $w=y^{\frac{4}{3}}$, then $w^3=y^4$, whence $3w^2dw=4y^3dy$, etc.]

10. Prove that $d(x^3-3x^2+6x-4)=3(x^2-2x+2)dx$, and that

$$d.(x^3-3x^2+6x-4)^4= 12(x-3x^2+6x-4)^3(x^2-2x+2)dx.$$

11. Prove that $d(ax+by)=adx+bdy$. Thence show that the characteristic $d$ of differentiation is distributive over a sum and commutative with a *constant factor*, just as if it were a number multiplier.

12. Prove that $d.\dfrac{y}{x}=\dfrac{xdy-ydx}{x^2}$. State this as a rule for differentiating fractions.

13. Prove that $d.\dfrac{y^2}{x^3}=\dfrac{y}{x^4}(2xdy-3ydx)$.

14. Prove that $d\dfrac{x^2+2xy+y^2}{(x+y)^2}=0$,

also that $d\,[(x+5)^2-x^2-10x]=0$.

15. If $m$, $n$ be any given positive integers, prove that

$$d.x^{\frac{\mathrm{m}}{\mathrm{n}}}=\frac{\mathrm{m}}{\mathrm{n}}x^{\frac{\mathrm{m}}{\mathrm{n}}-1}\,dx,\quad d.x^{-\frac{\mathrm{m}}{\mathrm{n}}}=-\frac{\mathrm{m}}{\mathrm{n}}x^{-\frac{\mathrm{m}}{\mathrm{n}}-1}\,dx.$$

State this result as a rule for differentiating powers to fixed fractional exponents.

[Let $y=x^{\frac{\mathrm{m}}{\mathrm{n}}}$, then $y^{\mathrm{n}}=x^{\mathrm{m}}$, $y'^{\mathrm{n}}=x'^{\mathrm{m}}$, and $y'^{\mathrm{n}}-y^{\mathrm{n}}=x'^{\mathrm{m}}-x^{\mathrm{m}}$, which may be written,

$$(y'^{\mathrm{n}-1}+y'^{\,\mathrm{n}-2}y+\ldots+y'y^{\mathrm{n}-2}+y^{\mathrm{n}-1})\;(y'-y)=$$
$$(x'^{\mathrm{m}-1}+x'^{\mathrm{m}-2}x+\ldots+x'x^{\mathrm{m}-2}+x^{\mathrm{m}-1})\;(x'-x).$$

Multipling this by $N$ and proceeding to the limits, $\lim x'=x$, $\lim y'=y$, $\lim N(x'-x)=dx$, $\lim N(y'-y)=dy$ we find $ny^{\mathrm{n}-1}dy=mx^{\mathrm{m}-1}dx$, and divided by $y^{\mathrm{n}}=x^{\mathrm{m}}$, this is $n\,dy\,/\,y=m\,dx\,/\,x$, or $dy=\dfrac{m}{n}\,y\dfrac{dx}{x}$ etc.]

16. Prove that $d.x^{\frac{4}{3}}y^{\frac{5}{4}}=\frac{4}{3}x^{\frac{1}{3}}y^{\frac{5}{4}}\,dx+\frac{5}{4}x^{\frac{4}{3}}y^{\frac{1}{4}}\,dy$.

17. Show that the successive derivatives of

$$x^5-7x^4+4x^3-9x^2+2x-7,$$

are $5x^4-28x^3+12x^2-18x+2$.

$20x^3-84x^2+24x-18$.

$60x^2-168x+24$.

$120x-168$,

$120$,

$0$.

18. Show that the successive derivatives of $(1+x)^4$ are $4(1+x)^3$, $12(1+x)^2$, $24(1+x)$, 24, 0.

19. Take various algebraic expressions and differentiate them by the full process, and also by rules, i. e., by examples already worked.

20. Expand $(1+x)^4$ by derivation.

[We know that $(1+x)^4=A+Bx+Cx^2+Dx^3+Ex^4$, for all values of $x$, where $A$, $B$, $C$, $D$, $E$, stand for some unknown numerical coefficients. Deriving this equation we find other identities,

$$4(1+x)^3=B+2Cx+3Dx^2+4Ex^3$$
$$12(1+x)^2=2C+6Dx+12Ex^2$$
$$24(1+x)=6D+24Ex$$
$$24=24E$$

Taking $x=o$ in these equations, since they are identities and so true for all values of $x$, we find $A=1$, $B=4$, $C=6$, $D=4$, $E=1$, and,

$$(1+x)^4=1+4x+6x^2+4x^3+x^4.]$$

21. Expand $(1+x)^n$ by derivation.

$$[(1+x)^n=1+nx+\frac{n(n-1)}{2\,!}x^2+\frac{n(n-1)\ (n-2)}{3\,!}x^3+..$$

where $2!=2.1$, $3!=3.2.1=6$, $4!=4.3.2.1=24$, etc.]

22. Expand $x^3-3x^2+2x-1$ in ascending powers of $x-4$, by derivation.

$[x^3-3x^2+2x-1=23+26(x-4)+9(x-4)^2+(x-4)^3]$.

# CHAPTER II

## Principles and Rules

**25.** PRINCIPLE 1. *If two variables are always equal, or if they always differ by a constant, then their differentials are always equal.*

For, let $x$, $y$ be original values of two variables and $x'$, $y'$ any new values as near as we please to the original values, then the conditions are, if the variables are always equal, that $y=x$ and $y'=x'$, or if the second always exceeds the first by a constant $c$, that $y=x+c$, $y'=x'+c$. In either case $y'-y=x'-x$, and therefore $N(y'-y)=N(x'-x)$, and as the new values are made to approach the old, while $N$ increases so that either member approaches a limit, the other member must approach the same limit, *i. e.*, $dy=dx$.

**26.** The proof of the above principle shows under what circumstances the differentiation of equals gives equals, *viz.*, the equation must remain true when the variables change from their original values by any corresponding amounts, however small. This principle is therefore not applicable to such an equation as $x^2-3x+2=0$, which is true for certain values of $x$ ($x=1$ or 2), but which does not remain true, when $x$ changes from those values. The equations to which the principle applies are of three classes, first, *absolute identities*, such as $(x+y)^2=x^2+2xy+y^2$; second, *limited* *identities* which are equal only for certain ranges of value of the variables, such as $1/(1-x)=1+x+x^2+x^3+$, etc., which is true only when $x$ is smaller than 1; and thirdly, equations that practically define one of the variables in terms of the others, such as $x^2+y^2=a^2$ which makes $y=\sqrt{(a^2-x^2)}$.

**27.** An alternative form of Principle 1 is that:

*The differential of a constant quantity is identically zero.*

For a constant can be made a function of any variables we please, as $2=x+2-x$, $1=x/x$, etc.; and as such a function, its change of value is zero; likewise any proportional change of value is zero, and hence the limit of such proportional change, or the required differential, is zero.

**28.** INVERSE PRINCIPLE 1. *If the differentials of two variables are always equal, then the variables are either always equal or always differ by a constant quantity.*

Two proofs of this will be given later, one geometric, and one algebraic. An alternative of this inverse principle is:

(*a*). *If the differential of a quantity is identically zero, then that quantity is a constant.*

**29.** PRINCIPLE 2. *The characteristic, d, of differentiation, is distributive over a sum, and commutative with a constant factor.* In symbols,

$$d(x+y)=dx+dy, \quad d.ax=a\,dx.$$

The proof will be left as an exercise. It is one of the first results the student would naturally notice in the practice of differentiation, and he would probably state it in some such form as, *the differential of a sum is the sum of the differentials of its terms*, and, *the differential of the product of a constant and a variable is the constant into the differential of the variable.* It is, however, important to consider it in the above form as a symbolic law of the characteristic $d$. The second part is really a consequence of the first, viz., $d.2x=d(x+x)=dx+dx=2\,dx$, etc.

**30.** By *the partial differentiation as to* $x$, of a function of two or more variables $x$, $y$, etc., we mean differentiation as if $x$ were the only variable, and the others were constants. The characteristics of partial differencing and differentiation as to $x$ will be $\Delta_x$, $d_x$, and as usual $d_x = \lim N\Delta_x$. Thus

$$\Delta_x(x^2y^3)=x'^2y^3-x^2y^3,\quad d_x(x^2y^3)=2\,xy^3dx.$$

Similarly,

$$\Delta_y(x^2y^3)=x^2y'^3-x^2y^3,\quad d_y(x^2y^3)=3x^2y^2dy.$$

**31.** A partial differential as to $x$ is simply a special value of the complete differential corresponding to any value of $dx$, and the values $dy=o$, $dz=o$, etc., since these are the values of $dy$, $dz$, that result by making $y$, $z$, constants. Thus if

$$d\,f(x,y,z)=f'(x,dx,y,dy,z,dz),$$

then

$$d_xf(x,y,z)=f'(x,dx,y,o,z,o),\ \text{etc.}$$

**32.** PRINCIPLE 3. *The complete differential of a func-*

*tion of several variables equals the sum of its partial differentials as to each variable.* In symbols,

$$df(x, y, z) = d_x f(x, y, z) + d_y f(x, y, z) + d_z f(x, y, z).$$

For let the complete change be made first by changing $x$ alone, then $y$ alone, then $z$ alone, giving the successive partial changes from $f(x,y,z)$ to $f(x',y,z)$ to $f(x',y',z)$ to $f(x',y',z')$ which are denoted by $\Delta_x f(x,y,z)$, $\Delta_y f(x',y,z)$, $\Delta_z f(x',y',z)$. The complete change of value of the function is easily seen to be the sum of these successive partial changes of value, i. e.,

$$\Delta f(x,y,z) = \Delta_x f(x,y,z) + \Delta_y f(x',y,z) + \Delta_z f(x',y',z).$$

Multiplying by $N$, we find an analagous result for the proportional differences, which is precisely the principle for differentials we wish to prove, except that the original values of the variables in the proportional differences are in the second, $x'$ instead of $x$, and in the third, $x'$, $y'$ instead of $x$, $y$. However, $x'$, $y'$, become $x$, $y$ in the limit, and the proportional differences become the differentials, so that if the general differential is a continuous function of its variables, $x'$, $y'$ will be replaced by $x$, $y$ in the differentials. For example in $N\Delta_y f(x',y,z)$ in which $x'$, $z$ are treated as constants, when $y'$ is very nearly $y$, this proportional difference is by definition, very nearly $d_y f(x',y,z)$, and this will be very nearly $d_y f(x,y,z)$ when $x'$ is very near $x$, if the latter differential is a continuous function, *i. e.*, if $df(x,y,z)$ is a continuous function (of which

$d_y f(x, y, z)$ is a special value obtained by making $dx=o$, $dz=o$.) Thus

$$df(x, y, z)=d_x f(x, y, z)+d_y f(x, y, z)+d_z f(x, y, z),$$

assuming, as is always the case in the calculus, that the functions considered are continuous, so that limits are found by substituting the limits of the variables.

**33.** That the limit of $fx'$ as $x'$ approaches $x$ is not always the same as $fx$, may be seen from the example $fx=x+$ integer part of $x$; taking $x=2$ and $x'$ less than and approaching 2, $fx'$ approaches, 3, but $fx=4$. This cannot occur when $fx$ is continuous, since then by definition, $fx'-fx$ approaches zero when $x'$ approaches $x$, and therefore $fx'$ approaches $fx$.

**34.** When $x, y, z$, are real variables, or, more generally, when $w=f(x, y, z)$ is an analytical function of its variables whether they are real or imaginary, then by Article 21 $d_x w / dx$ is independent of $dx$, and therefore a function of $x, y, z$ alone. This quotient is called *the partial derivative of $w$ as to $x$*, and is denoted by $\partial w / \partial x$, the script d being notice of partial differentation, while the denominator shows the variable of differentiation. In this notation, we have.

$$\text{(a).}\quad dw=\frac{\partial w}{\partial x}dx+\frac{\partial w}{\partial y}dy+\frac{\partial w}{\partial z}dz$$
$$=f_1(x, y, z)\,dx+f_2(x, y, z)dy+f_3(x, y, z)dz.$$

**35.** The use of Principle 3 greatly simplifies the differentiation of many complicated expressions. In the first place it reduces differentiation to the consideration

of one variable at a time, and secondly, an expression involving one variable only may be made a function of several variables by replacing selected component parts of the expression by new letters for the time being; thence differentiating as to the several variables and adding, we find the differential of the whole. Thus

$$d.x^2 = d.xy, \text{ where } y = x, = d_x.xy + d_y.xy$$
$$= y\,dx + x\,dy = xdx + xdx = 2\,xdx;$$
$$d.x^3 = d.x.x.x = x^2dx + x^2dx + x^2dx = 3\,x^2dx;$$

and in general, if $n$ be any positive integer, then,

$$d.x^n = d.x.x\ldots \text{ to } n \text{ factors} = x^{n-1}\,dx + x^{n-1}\,dx + \ldots$$
$$\text{to } n \text{ terms} = nx^{n-1}\,dx.$$

It is not necessary to replace each component expression that is fixed upon as a single variable by a new letter, since a little practice in retaining the idea of its singleness of value as distinct from other such single values into which the expression may be conceived as separated, will accomplish the same purpose, and be shorter and easier. The above differentiations of $x^3$ and $x^n$ are examples of this, in which each factor $x$ is conceived as distinct from every other such factor for the purposes of partial differentiation as to that factor, without the necessity of displacing it by another letter for the time being.

## Powers

**36.** Rule 1. *To differentiate a power with a constant exponent, multiply the power by its exponent and the differ-*

*ntial of its base, and divide the result by the base.* In ymbols

$$d.x^p = p.x^p dx/x = px^{p-1} dx.$$

The proof of this rule for any real exponent $p$ be made to depend upon the theorem that $(y^p-1)/(y-1)=p$ when $\lim y=1$. This is proved when $p$ is a fraction, $\pm m/n$, by dividing out the common factor $y^{\pm\frac{1}{n}}-1$ from numerator and denominator before putting $y=1$; then if $p$ is an incommensurable number between the fractions $q$, $q'$, in order of magnitude, the quotient $(y^p-1)/(y-1)$ will lie between the similar quotients found by replacing $p$ by $q$ and $q'$, and its limit will therefore be between $q$ and $q'$. Thus since this limit *always* lies between the same two fractions as $p$ does, it must be $p$. Hence

$$N\Delta.x^p = N(x'^p - x^p) = \frac{(y^p-1)x^p}{(y-1)x} N\Delta x,$$

where $y=x'/x$, and therefore approaches 1 as $x'$ approaches $x$. Thus

$$d.x^p = \lim N\Delta.x^p = px^p dx/x = px^{p-1} dx.$$

To prove this rule for imaginary exponents, it must be necessary to define such powers. It might be made a condition of that definition that the above rule should be true, and this would, in fact, be sufficient to determine such powers in connection with the condition that $1^p=1$.

**37.** GENERALIZED RULE 1. *To differentiate a product of powers, each with a constant exponent, multiply the pro-*

*tion of several variables equals the sum of its partial differentials as to each variable.* In symbols,

$$df(x, y, z) = d_x f(x, y, z) + d_y f(x, y, z) + d_z f(x, y, z).$$

For let the complete change be made first by changing $x$ alone, then $y$ alone, then $z$ alone, giving the successive partial changes from $f(x, y, z)$ to $f(x', y, z)$ to $f(x', y', z)$ to $f(x', y', z')$ which are denoted by $\Delta_x f(x, y, z)$, $\Delta_y f(x', y, z)$, $\Delta_z f(x', y', z)$. The complete change of value of the function is easily seen to be the sum of these successive partial changes of value, i. e.,

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$d_y f(x, y, z)$ is a special value obtained by making $dx=o$, $dz=o$.) Thus

$$df(x, y, z)=d_x f(x, y, z)+d_y f(x, y, z)+d_z f(x, y, z),$$

assuming, as is always the case in the calculus, that the functions considered are continuous, so that limits are found by substituting the limits of the variables.

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$$=f_1(x, y, z)\,dx+f_2(x, y, z)dy+f_3(x, y, z)dz.$$

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*tion of several variables equals the sum of its partial differentials as to each variable.* In symbols,

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For let the complete change be made first by changing $x$ alone, then $y$ alone, then $z$ alone, giving the successive partial changes from $f(x,y,z)$ to $f(x',y,z)$ to $f(x',y',z)$ to $f(x',y',z')$ which are denoted by $\Delta_x f(x,y,z)$, $\Delta_y f(x',y,z)$, $\Delta_z f(x',y',z)$. The complete change of value of the function is easily seen to be the sum of these successive partial changes of value, i. e.,

$$\Delta f(x,y,z) = \Delta_x f(x,y,z) + \Delta_y f(x',y,z) + \Delta_z f(x',y',z).$$

Multiplying by $N$, we find an analagous result for the proportional differences, which is precisely the principle for differentials we wish to prove, except that the original values of the variables in the proportional differences are in the second, $x'$ instead of $x$, and in the third, $x'$, $y'$ instead of $x$, $y$. However, $x'$, $y'$, become $x$, $y$ in the limit, and the proportional differences become the differentials, so that if the general differential is a continuous function of its variables, $x'$, $y'$ will be replaced by $x$, $y$ in the differentials. For example in $N\Delta_y f(x',y,z)$ in which $x'$, $z$ are treated as constants, when $y'$ is very nearly $y$, this proportional difference is by definition, very nearly $d_y f(x',y,z)$, and this will be very nearly $d_y f(x,y,z)$ when $x'$ is very near $x$, if the latter differential is a continuous function, *i. e.*, if $df(x,y,z)$ is a continuous function (of which

$d_y f(x, y, z)$ is a special value obtained by making $dx = o$, $dz = o$.) Thus

$$df(x, y, z) = d_x f(x, y, z) + d_y f(x, y, z) + d_z f(x, y, z),$$

assuming, as is always the case in the calculus, that the functions considered are continuous, so that limits are found by substituting the limits of the variables.

**33.** That the limit of $fx'$ as $x'$ approaches $x$ is not always the same as $fx$, may be seen from the example $fx = x +$ integer part of $x$; taking $x = 2$ and $x'$ less than and approaching 2, $fx'$ approaches, 3, but $fx = 4$. This cannot occur when $fx$ is continuous, since then by definition, $fx' - fx$ approaches zero when $x'$ approaches $x$, and therefore $fx'$ approaches $fx$.

**34.** When $x, y, z$, are real variables, or, more generally, when $w = f(x, y, z)$ is an analytical function of its variables whether they are real or imaginary, then by Article 21 $d_x w / dx$ is independent of $dx$, and therefore a function of $x, y, z$ alone. This quotient is called *the partial derivative of $w$ as to $x$*, and is denoted by $\partial w / \partial x$, the script d being notice of partial differentation, while the denominator shows the variable of differentiation. In this notation, we have.

$$\text{(a).} \quad dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$
$$= f_1(x, y, z)\, dx + f_2(x, y, z) dy + f_3(x, y, z) dz.$$

**35.** The use of Principle 3 greatly simplifies the differentiation of many complicated expressions. In the first place it reduces differentiation to the consideration

*tion of several variables equals the sum of its partial differentials as to each variable.* In symbols,

$$df(x, y, z) = d_x f(x, y, z) + d_y f(x, y, z) + d_z f(x, y, z).$$

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**35.** The use of Principle 3 greatly simplifies the differentiation of many complicated expressions. In the first place it reduces differentiation to the consideration

$$\log\left\{x+\sqrt{(x^2\pm a^2)}\right\},\quad \frac{1}{a}\log\frac{x}{a+\sqrt{(a^2\pm x^2)}},$$

$$\frac{1}{2a}\log\frac{x-a}{x+a}.$$

**54.** On the contrary the differentials (similar to the above),

$$\frac{dx}{\sqrt{(a^2-x^2)}},\quad \frac{dx}{x\sqrt{(x^2-a^2)}}.\quad \frac{dx}{x^2+a^2},$$

are anti-trigonometric forms, their integrals being,

$$\sin^{-1}\frac{x}{a},\quad \frac{1}{a}\sec^{-1}\frac{x}{a},\quad \frac{1}{a}\tan^{-1}\frac{x}{a}.$$

The anti-vers form, $\dfrac{dx}{\sqrt{(2ax-x^2)}}=d.\operatorname{vers}^{-1}\dfrac{x}{a}$ is also the anti-sin form $d.\sin^{-1}\dfrac{x-a}{a}$. In fact the two integrals differ by the constant $\pi/2$.

**55.** Other logarithmic differentials are,

$$\sec x\,dx,\ \csc x\,dx,\ \tan x\,dx,\ \cot x\,dx=\cos x\,dx/\sin x.$$

By multiplying and dividing the first three by

$$\sec x+\tan x,\ \csc x-\cot x,\ \sec x,$$

respectively, the integrals are seen to be,

$$\log(\sec x+\tan x),\ \log(\csc x-\cot x),\ \log\sec x,\ \log\sin x.$$

**56.** The differential $\dfrac{dx}{a\sin x+b\cos x+c}$, reduces to the form $\frac{c' dz}{z^2 \pm c^2}$, according to the values of $a$, $b$, $c$, by the transformation $z = \tan \frac{1}{2} x$, and therefore $dz = \frac{1}{2} \sec \frac{1}{2} x^2 \, dx$. To make the transformation, first put $\sin x = 2 \sin \frac{1}{2} x \cos \frac{1}{2} x$, $\cos x = \cos \frac{1}{2} x^2 - \sin \frac{1}{2} x^2$, $b = b (\cos \frac{1}{2} x^2 + \sin \frac{1}{2} x^2)$ and multiply both numerator and denominator by $\sec \frac{1}{2} x^2$.

**57.** The integral $\int \frac{dz}{(z^2 + c^2)^n}$, where $n$ is a positive integer is reduced ultimately to finding $\int \frac{c' dz}{z^2 + c^2}$, by successive applications of the formula,

$$\text{(a)} \quad \int \frac{dz}{(z^2+c^2)^n} = \frac{1}{2(n-1)c^2} \left\{ \frac{z}{(z^2+c^2)^{n-1}} + (2n-3) \int \frac{dz}{(z^2+c^2)^{n-1}} \right\}$$

This formula may be verified by differentiation.

**58.** It can now be shown how to find $\int \frac{f_m x}{f_n x} dx$, where $f_m x$, $f_n x$ are entire functions of $x$, of degrees $m$, $n$, respectively, with real coefficients. If $m$ is not less than $n$, we divide the numerator by the denominator to a remainder of less degree than $n$. The entire part of the quotient is integrated term by term under Inverse Rule 1. We have therefore only to consider a proper fraction of the above form, *i.e.*, one in which $m < n$. By the theory of equations, the denominator $f_n x$ factors into real irreducible or prime factors, of the forms $x - a$, or $(x-b)^2 + c^2$, each occurring to certain powers. By the theory of resolution into partial fractions, the given proper fraction $f_m x / f_n x$ will reduce to a sum of fractions each involving a power of *one prime factor only* in its denominator, and a numerator *one degree less than the prime factor of the denominator*, provided, *all fractions of this form be included in the sum whose denominators are divisors of* $f_n x$. The integration of any such partial fractions comes under preceding methods. The factoring of the denominator, and the resolution of the fraction into the sum of its partial fractions, are algebraic problems.

**59.** The Exponential Differential, $d.ca^x = c \log a.\ a^x dx$. This differential consists of a power with constant base, multiplied, to a constant, by the differential of the exponent. It is anti-differentiated by *dividing it by the product of the differential of the exponent and the natural logarithm of the base.*

**60.** The Trigonometric Differentials, $c \sin x\, dx$, $c \cos x\, dx$, $c \sec x^2\, dx$, $c \csc x^2\, dx$, $c \sec x \tan x\, dx$, $c \csc x \cot x\, dx$. These consist of certain trigonometric functions of a variable, multiplied, to a constant, by the differential of the variable. The integration consists in replacing these functions in each differential by the corresponding function of the same variable from which it is derived and dividing the result by the differential of the variable, and also by $-1$ if the resulting function is a "co" function. Thus, the integrals are respectively, $-c \cos x$, $c \sin x$, $c \tan x$, $-c \cot x$, $c \sec x$, $-c \csc x$.

**61.** Integration By Parts. Principle 3 can be reversed in integration. Its particular application is to a differential $u\,dv$, where $u$ is a chosen variable factor and $dv$ a known differential. From $d(uv)=u\,dv+v\,du$, we find $\int u\,dv=uv-\int v\,du$. In words, *to integrate by parts, integrate as if a chosen variable factor were a constant for the first term, and complete the integration by subtracting the complete integral of the differential of the first term as if the assumed constant alone varied.* The success of this method, when it is applicable, depends upon the choice of the variable factor, which must be such that the new integral to be found is easier of solution than the given one.

E.g.
$$\begin{aligned}\int x^2\,e^x\,dx &= x^2 e^x - 2\int x\,e^x\,dx\\ &= x^2\,e^x - 2x\,e^x + 2\int e^x\,dx\\ &= x^2\,e^x - 2x\,e^x + 2\,e^x.\end{aligned}$$

This is by taking $e^x dx = d.e^x$ as the known differential each time, and the remaining factor as a constant in the partial integration. But if we take $e^x$ as the assumed constant, we find

$$\int x^2\,e^x\,dx = \frac{x^3}{3}e^x - \tfrac{1}{3}\int x^3\,e^x\,dx,$$

and the new integral is more difficult than the old.

**62.** In the following examples that give a differential, followed by one or more of its integrals, it is required: first to verify the differentiation by the principles and rules of preceding articles, as an exercise in differentiation; secondly, to obtain the integral from the differential by the inverse rules or methods of the preceding forms, as an exercise in such methods and rules. The more important integrations may be taken as fundamental forms in any subsequent examples of integration.

EXAMPLES.

| DIFFERENTIALS. | INTEGRALS. |
|---|---|
| 1. $(8x^3-9x^2+6x-7)\,dx;$ | $2x^4-3x^3+3x^2-7x+8$ |
| 2. $(3\sqrt{x}+\frac{6}{\sqrt{x}}-9x^2+\frac{2}{x^3})dx;$ | $2x^{\frac{3}{2}}+12\sqrt{x}-3x^3-x^{-2}$ |
| 3. $(4x-1)dx;$ | $2x^2-x+3,\ (4x-1)^2/8.$ |
| 4. $(4-3x)^2dx;$ | $3+16x-12x^2+3x^3,\ -(4-3x)^3/9$ |
| 5. $\sqrt{(4a+9x)}\,dx;$ | $2(4a+9x)^{\frac{3}{2}}/27$ |
| 6. $s\,ds/\sqrt{(a^2+s^2)};$ | $\sqrt{(a^2+s^2)}.$ |
| 7. $x^{n-1}(a+bx^n)^h\,dx;$ | $(a+bx^n)^{h+1}/bn(h+1)$ |
| 8. $x^m(a+bx^n)^h\{(m+1)a+(m+nh+n+1)bx^n\}dx;$ | $x^{m+1}(a+bx^n)^{h+1}$ |
| 9. $x^{-n(h+1)-1}(a+bx^n)^h\,dx;$ | $-x^{-n(h+1)}(a+bx^n)^{h+1}/an(h+1)$ |
| 10. $x^2\,dx/(a^2+cx^2)^{\frac{5}{2}};$ | $x^3/(a^2+cx^2)^{\frac{3}{2}}3a^2$ |
| 11. $x^h\,dx/(a^2+cx^2)^{\frac{h+3}{2}};$ | $x^{h+1}/(a^2+cx^2)^{\frac{h+1}{2}}a^2(h+1)$ |
| 12. $x(a^2+cx^2)^{\frac{h}{2}}\,dx;$ | $(a^2+cx^2)^{\frac{h+2}{2}}/c(h+2)$ |

13. $\sqrt{(a^2+cx^2)}dx=\frac{\sqrt{(a^2+cx^2)}}{x}\,.\,x\,dx$, for integration by parts; $\frac{1}{2}x\sqrt{(a^2+cx^2)}+\frac{a^2}{2}\int\frac{dx}{\sqrt{(a^2+cx^2)}}$

14. $\sqrt{(a^2-x^2)}\,dx$; $\qquad \frac{1}{2}x\sqrt{(a^2-x^2)}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$

15. $\sqrt{(a^2+x^2)}\,dx$; $\frac{1}{2}x\sqrt{(a^2+x^2)}+\frac{a^2}{2}\log(x+\sqrt{a^2+x^2})$

16. $(a^2-x^2)^{\frac{3}{2}}\,dx=\frac{(a^2-x^2)^{\frac{3}{2}}}{x^3}\cdot x^3\,dx$, for integration by parts;

$$\frac{x}{8}(5a^2-2x^2)\sqrt{(a^2-x^2)}+\frac{3a^4}{8}\sin^{-1}\frac{x}{a}$$

17. $(a^2+x^2)^{\frac{3}{2}}dx$;

$$\frac{x}{8}(5a^2+2x^2)\sqrt{(a^2+x^2)}+\frac{3a^4}{8}\log(x+\sqrt{a^2+x^2})$$

18. $\frac{xdx}{\sqrt{(2ax-x^2)}}$; $\qquad -\sqrt{(2ax-x^2)}+a\,\text{vers}^{-1}\frac{x}{a}$

19. $\frac{dx}{x\sqrt{(2ax-x^2)}}$; $\qquad -\frac{\sqrt{(2ax-x^2)}}{ax}$

20. $\sqrt{(2ax-x^2)}\,dx$; $\frac{x-a}{2}\sqrt{(2ax-x^2)}+\frac{a^2}{2}\text{vers}^{-1}\frac{x}{a}$

21. $\frac{dx}{x^3\sqrt{(a^2-x^2)}}$;

$$-\frac{\sqrt{(a^2-x^2)}}{2a^2x^2}+\frac{1}{2a^3}\log\frac{x}{a+\sqrt{(a^2-x^2)}}$$

22. $\frac{dx}{x^3\sqrt{(x^2-a^2)}}$; $\qquad \frac{\sqrt{(x^2-a^2)}}{2a^2x^2}+\frac{1}{2a^3}\sec^{-1}\frac{x}{a}$

23. $\frac{x\,dx}{a^2+cx^2}$; $\qquad \frac{1}{c}\log\sqrt{(a^2+cx^2)}$

24. $\frac{dx}{x(a+bx^n)}$; $-\frac{1}{an}\log(ax^{-n}+b)$, $\frac{1}{an}\log\frac{x^n}{a+bx^n}$

25. $\dfrac{1+x^2}{x-x^3}\,dx$; $\log\dfrac{x}{1-x^2}$

26. $\dfrac{x^3+1}{x(x-1)^3}\,dx$; $\log\dfrac{(x-1)^2}{x}-\dfrac{x}{(x-1)^2}$

27. $\dfrac{x^3-1}{x^3+3x}\,dx$; $x+\frac{1}{6}\log\dfrac{x^2+3}{x^2}-\sqrt{3}\tan^{-1}\dfrac{x}{\sqrt{3}}$

28. $\dfrac{dx}{x^4+1}$; $\dfrac{1}{4\sqrt{2}}\log\dfrac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\dfrac{1}{2\sqrt{2}}\tan^{-1}\dfrac{x\sqrt{2}}{1-x^2}$

29. $\dfrac{x^3+x-1}{(x^2+2)^2}\,dx$;

$$\tfrac{1}{2}\log(x^2+2)-\frac{1}{4\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}-\frac{x-2}{4(x^2+2)}$$

30. $dx/x\log x$; $\log^2 x=\log\log x$

31. $\log x\,dx/x$; $\log x^2/2=\log x.\log x/2$

32. $2x\,dx/x^2$; $\log.x^2$

33. $\dfrac{dx}{\log x}$; $\displaystyle\int\frac{dx}{\log x}$ (a function not tabulated)

34. $\sqrt{\dfrac{a+x}{b+x}}\,dx$;

$$\sqrt{(a+x)(b+x)}+(a-b)\log(\sqrt{a+x}+\sqrt{b+x})$$
[Put $b+x=y^2$, $dx=2y\,dy$]

35. $\sqrt{\dfrac{a-x}{b+x}}.\,dx$; $\sqrt{(a-x)(b+x)}+(a+b)\sin^{-1}\sqrt{\dfrac{x+b}{a+b}}$

36. $(e^x+e^{-x})^2\,dx$; $\frac{1}{2}(e^{2x}-e^{-2x}+4x)$

37. $\dfrac{e^x-1}{e^x+1}\,dx$; $2\log(e^x+1)-x$

38. $a^x b^x\,dx$; $a^x b^x/\log ab$

39. $\dfrac{e^{3x}\,dx}{e^x-1}$; $\qquad \frac{1}{2}e^{2x}+e^x+\log(e^x-1)$

40. $x^2\,e^x\,dx$; $\qquad e^x(x^2-2x+2)$ (int. by parts)

41. $x^2 \log x\,dx$; $\qquad \frac{1}{9}x^3(3\log x-1)$

42. $\sin^{-1}x\,dx$; $\qquad x\sin^{-1}x+\sqrt{(1-x^2)}$

43. $\sin 2\,x.dx$; $\qquad -\frac{1}{2}\cos 2x$

44. $\cos x^2\,dx$; $\qquad \frac{1}{2}x+\frac{1}{4}\sin 2x$

45. $\cos x^3\,dx$; $\qquad \sin x-\frac{1}{3}\sin x^3$

46. $(\tan x+\cot x)^2\,dx$; $\qquad \tan x-\cot x$

47. $(\tan 2x-1)^2\,dx$; $\qquad \frac{1}{2}\tan 2x+\log\cos 2x$

48. $e^{mx}\sin nx dx$; $\qquad e^{mx}(m\sin nx-n\cos nx)/(m^2+n^2)$

[Integrate twice by parts with $e^{mx}$ as constant]

49. $e^{mx}\cos nx\,dx$; $\qquad e^{mx}(n\sin nx+m\cos nx)/(m^2+n^2)$

50. $\tan x^2\,dx$; $\qquad \tan x-x$

51. $\tan x^3\,dx=\sec x^2\tan x\,dx-\tan x\,dx$;

$\frac{1}{2}\tan x^2+\log\cos x$

52. $$\int \tan x^n\,dx=\frac{\tan x^{n-1}}{n-1}-\int \tan x^{n-2}\,dx$$

53. $\tan x^{2n}\,dx$, $n$ a positive integer;

$$\frac{\tan x^{2n-1}}{2n-1}-\frac{\tan x^{2n-3}}{2n-3}+\ldots+(-1)^{n+1}(\tan x-x)$$

54. $\tan x^{2n-1}\,dx$, $n$ a positive integer;

$$\frac{\tan x^{2n-2}}{2n-2}-\frac{\tan x^{2n-4}}{2n-4}+\ldots+(-1)^n\left(\frac{\tan x^2}{2}+\log\cos x\right)$$

55. $\sec x^4\,dx=\sec x^2(1+\tan x^2)dx$; $\qquad \tan x+\frac{1}{3}\tan x^3$

56. $\cos x^4\, dx = \frac{1}{4}(1+\cos 2x)^2\, dx$;

$$\tfrac{1}{32}(12x+8\sin 2x+\sin 4x)$$

57. $\cos x^4 \sin x^2\, dx = \frac{1}{8}(1+\cos 2x)(\sin 2x)^2\, dx$

$= \frac{1}{16}(1-\cos 4x)\, dx + \frac{1}{8}\sin 2x^2 \cos 2x\, dx$;

$$\frac{x}{16} - \frac{\sin 4x}{64} + \frac{1}{48}\sin 2x^3$$

58. $\cos x^4 \sin x^3\, dx = \cos x^4 (1-\cos x^2)\sin x\, dx$;

$$-\tfrac{1}{5}\cos x^5 + \tfrac{1}{7}\cos x^7$$

59. $\dfrac{dx}{5-4\cos 2x}$; $\qquad \frac{1}{3}\tan^{-1}(3\tan x)$

60. If $\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

then $\cosh x^2 - \sinh x^2 = 1$

$d \sinh x = \cosh x\, dx$

$d \cosh x = \sinh x\, dx$

$d \sinh^{-1} x = dx / \surd(x^2+a^2)$

$d \cosh^{-1} x = dx / \surd(x^2-a^2)$

**63. Reduction Formulas**

Let $v = \surd(a^2+cx^2)$; then we have the differential rules,

$$d.\, v^n = c\, n\, v^{n-2} x dx$$

$$d\left(\frac{x}{v}\right)^n = a^2 n \left(\frac{x}{v}\right)^{n-1}\frac{dx}{v^3}.$$

These rules may be used to integrate $x^m v^n dx$ by parts in six different ways, so that the new integral shall be $\int x^{m'} v^{n'} dx$ where $m'$, $n'$ are one or both two units smaller than $m$, $n$; and repeated applications of such integrations will theretore reduce the given integral eventually to dependence upon standard forms, either algebraic, logarithmic or anti-trigonometric, when $m$, $n$ are any integers positive or negative. These formulas are,

$$\int x^m v^n\, dx =$$

(a) $\int v^n \,.\, x^m dx$

$$= \frac{1}{m+1}\left\{ x^{m+1} v^n - cn \int x^{m+2} v^{n-2}\, dx \right\}$$

(b) $\int x^{m-1} \,.\, xv^n \;\, dx$

$$= \frac{1}{c(n+2)}\left\{ x^{m-1} v^{n+2} - (m-1)\int x^{m-2} v^{n+2}\, dx \right\}$$

(c) $\int \frac{v^n}{x^n} \,.\, x^{m+n}\, dx$

$$= \frac{1}{m+n+1}\left\{ x^{m+1} v^n + a^2 n \int x^m v^{n-2}\, dx \right\}$$

(d) $\int x^{m+n+3} \,.\, \frac{v^n}{x^{n+3}}\, dx$

$$= \frac{1}{a^2(n+2)}\left\{ -x^{m+1} v^{n+2} + (m+n+3)\int x^m v^{n+2}\, dx \right\}$$

(e) $\int \frac{x^{m-1}}{v^{m-1}} \,.\, x\, v^{m+n-1}\, dx$

$$= \frac{1}{c(m+n+1)}\left\{ x^{m-1} v^{n+2} - a^2(m-1)\int x^{m-2} v^n\, dx \right\}$$

(f) $\int v^{m+n+3} \,.\, \frac{x^m\, dx}{v^{m+3}}$

$$= \frac{1}{a^2(m+1)}\left\{ x^{m+1} v^{n+2} - c(m+n+3)\int x^{m+2} v^n\, dx \right\}$$

In these formulas, $a^2$ may be changed to $-a^2$ throughout, and $c$ is usually 1 or $-1$.

Similarly, let $u=\sin x$, $v=\cos x$, so that

$$d.u^n=n\,u^{n-1}v\,dx,$$
$$d.v^n=-n\,v^{n-1}u\,dx$$

$$d.\left(\frac{u}{v}\right)^n=n.\frac{u^{n-1}\,dx}{v^{n+1}},\qquad d\left(\frac{v}{u}\right)^n=-n\frac{v^{n-1}\,dx}{u^{n+1}}$$

and then

$$\int u^m v^n\,dx=$$

(a′) $\int v^{n-1}.u^m v\,dx$

$$=\frac{1}{m+1}\left\{u^{m+1}v^{n-1}+(n-1)\int u^{m+2}v^{n-2}\,dx\right\}$$

(b′) $\int u^{m-1}.v^n u\,dx$

$$=\frac{1}{n+1}\left\{-u^{m-1}v^{n+1}+(m-1)\int u^{m-2}v^{n+2}\,dx\right\}$$

(c′) $\int \frac{v^{n-1}}{u^{n-1}}.u^{m+n-1}v\,dx$

$$=\frac{1}{m+n}\left\{u^{m+1}v^{n-1}+(n-1)\int u^m v^{n-2}\,dx\right\}$$

(d′) $\int u^{m+n+2}.\frac{v^n\,dx}{u^{n+2}}$

$$=\frac{1}{n+1}\left\{-u^{m+1}v^{n+1}+(m+n+2)\int u^m v^{n+2}\,dx\right\}$$

(e′) $\int \frac{u^{m-1}}{v^{m-1}}.v^{m+n-1}u\,dx$

$$=\frac{1}{m+n}\left\{-u^{m-1}\,v^{n+1}+(m-1)\int u^{m-2}\,v^{n}\,dx\right\}$$

(f′) $\int v^{m+n+2}.\ \frac{u^m\,dx}{v^{m+2}}$

$$=\frac{1}{m+1}\left\{u^{m+1}v^{n+1}+(m+n+2)\int u^{m+2}\,v^{n}\,dx\right\}$$

61. $\sin x^6\,dx=\tan x^5.\cos x^5\sin x\,dx;$

$$-\tfrac{1}{6}\tan x^5\cos x^6+\tfrac{5}{6}\int\tan x^3\cos x^3\sin x\,dx,$$

$$-\tfrac{1}{6}\tan x^5\cos x^6-\tfrac{5}{24}\tan x^3\cos x^4+\tfrac{5}{8}\int\tan x\cos x\sin x\,dx,$$

$$-\tfrac{1}{6}\sin x^5\cos x-\tfrac{5}{24}\sin x^3\cos x-\tfrac{5}{16}\sin x\cos x+\frac{5x}{16}.$$

62. $\cos x^4\sin x^2\,dx;$

$$\frac{\cos x\ \sin x}{48}(3+2\cos x^2-8\cos x^4)+\frac{x}{16}$$

63. $\frac{dx}{\sin x^4\cos x^3};-\frac{1}{\cos x^2}\left(\frac{1}{3\sin x^3}+\frac{5}{3\sin x}-\tfrac{5}{2}\sin x\right)$

$$+\tfrac{5}{2}\log(\sec x+\tan x)$$

## 64. Indeterminate Forms

RULE. *To evaluate* $\frac{fx}{Fx}$ *for a given value of x that makes* $fx=o$, $Fx=o$, *differentiate both numerator and denominator, before substituting the given value of x; and similarly for values of x that make* $fx=\infty$, $Fx=\infty$. *Before making such differentiations, any factor of* $f(x)/Fx$ *that is not zero or infinity for the given value of x may be replaced by its value for such value of x.*

For let $fa=o$, $Fa=o$, then :

$$\frac{fa}{Fa}=\lim_{x'=a}\frac{fx'}{Fx'}=\lim_{x'=a}\frac{fx'-fa}{Fx'-Fa}=\lim\frac{\Delta fa}{\Delta Fa}=\frac{dfa}{dFa}.$$

If $fa=\infty$, $Fa=\infty$, then $1/fa=o$, $1/Fa=o$, and

$$b=\frac{fa}{Fa}=\lim\frac{1}{Fx}\Big/\frac{1}{fx}=\lim\frac{dFx}{Fx^2}\Big/\frac{dfx}{fx^2}$$

$=b^2\, dFa/d\,fa$; *i.e.*, $b=d\,fa/d\,Fa$.

Finally, any factor whose limit is finite, can, by the principle that the limit of a product equals the product of the limits of its factors, be at once replaced by its limit, and the limit of the remaining factor may be found by itself.

Exponential indeterminate forms must be evaluated through their logarithms. E.g. $y=(1+\frac{1}{x})^x$ when $x=\infty$ whose form is $1^\infty$, must be evaluated from $\log y=x\log(1+\frac{1}{x})=\log(1+z)/z$ where $z=\frac{1}{x}=o$. This is $o/o$, and is therefore when $z=o$, $d\log(1+z)/dz$ $=\frac{1}{1+z}=1$; $\log y=1$, $y=e$.

64. Evaluate the following functions for the given values of $x$:

(a) $\dfrac{x^{\sqrt{2}}-1}{x-1}$, $x=1$ ($\sqrt{2}$)

(b) $\dfrac{x^3-x^2-x+1}{x^3+x^2-5x+3}$, $x=1$ ($\frac{1}{2}$)

(c) $\dfrac{x-5}{(x-4)^2-1}$, $x=5$ $(\frac{1}{4})$

(d) $\log x/(x-1)$, $x=1$ $(1)$

(e) $(e^x+e^{-x})/x$, $x=o$ $(2)$

(f) $\dfrac{x\sin x-x^2}{2\cos x+x^2-2}$, $x=o$ $(-2)$

(g) $\log\sin x/\cos x$, $x=\pi/2$ $(0)$

(h) $\sin^{-1}x/\sin x$, $x=o$ $(1)$

(i) $\tan x/\log\cos x$, $x=\pi/2$ $(\infty)$

(j) $e^x(\cos x-1)/x\log(1+x)$, $x=o$ $(-\frac{1}{2})$

[Note that the factor $e^x$ can be replaced by $e^0=1$ *before* differentiating numerator and denominator].

(k) $\sec x-\tan x$, $x=\pi/2$ $(0)$

(l) $(1+x^2)^{\frac{1}{x}}$, $x=o$ $(1)$

(m) $(1+x)^{\pm\frac{1}{x^2}}$, $x=o$ $(\infty, 0)$

(n) $(1+mx)^{\frac{1}{x}}$, $x=o$ $(e^m)$

(o) $(\log x)^x$, $x=o$ $(1)$

(p) $x^{\frac{1}{\log(1+x)}}$, $x=\infty$ $(e)$

(q) $\dfrac{(x+1)^2\log(2x^2-1).^2}{\tan(x-1).^2}$, $x=1$ $(64)$

## 65. Applications of Inverse Principle 1

65. If $x\,dx+y\,dy=o$ and $y=a$ when $x=o$, then $x^2+y^2=a^2$

66. If $\frac{x\,dx}{a^2}+\frac{y\,dy}{b^2}=o$ and $y=b$ when $x=o$,

then $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

67. If $\frac{dx}{x^{\frac{1}{3}}}+\frac{dy}{y^{\frac{1}{3}}}=o$, and $y=a$ when $x=o$,

then $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$

68. If $\frac{dy}{dx}=\frac{2a}{y}$, and $y=o$ when $x=o$, then $y^2=4a\,x$

69. If $\frac{r\,d\theta}{dr}=c$, and $r=a$ when $\theta=o$, then $r=a\,e^{\frac{\theta}{c}}$

70. If $(1+x)^3=1+3x+3x^2+x^3$, then by integration,

$(1+x)^4=1+4x+6x^2+4x^3+x^4$

71. From $\frac{1}{1-x^2}=1+x^2+x^4+x^6+\ldots$, $(x^2<1)$

show that $\log\frac{1+x}{1-x}=2\,(x+\frac{1}{3}x^3+\frac{1}{5}x^5+\ldots)$,

72. From $\frac{1}{\sqrt{(1-x^2)}}$

$$=1+\tfrac{1}{2}x^2+\frac{1.3}{2.4}x^4+\ldots+\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}x^{2n}+\ldots$$

when $x^2<1$, show that for the same values of $x$, $\sin^{-1}x=$

$$x+\tfrac{1}{2}\,\frac{x^3}{3}+\frac{1.3}{2.4}\,\frac{x^5}{5}+\ldots+\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}\,\frac{x^{2n+1}}{2n+1}+\ldots$$

73. From $\frac{1}{1+x^2}=1-x^2+x^4-x^6+\ldots$ $(x^2<1)$

show that, $\tan^{-1}x=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\ldots$

and that, $\tan^{-1}\frac{1+x}{1-x}=\frac{\pi}{4}+x-\frac{1}{3}x^3+\frac{1}{5}x^5-\ldots$

74. Show that the function $e^x$ *defined* by the differential equation $d.e^x = e^x.dx$, and the initial value $e^o = 1$, satisfies the exponent law $e^x.e^y = e^{x+y}$, and that this function is $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ (a convergent series for all values of $x$).

[Prove the ratio of the two members constant, by differentiating such ratio, and determine the constant by its value for $x = o$].

NOTE. $e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots = 2.71828\dots = e$; also by the exponential law, $e^2 = e^1.e^1 = e.e$, $e^3 = e^2.e^1 = e.e.e$, $e^4 = e^3 e^1 = e.e.e.e$, etc. Thus the exponent notation $e^x$ for $1 + x + \frac{x^2}{2!} + \dots$ agrees with the usual meaning of this notation when $x$ is a positive integer, and it must agree also when $x$ is any real number, since the generalization of the exponent in elementary algebra, is derived from the exponential law, $a^m.a^n = a^{m+n}$. The above generalization extends also to imaginary exponents.

75. If $i = \sqrt{-1}$, show that $e^{ix} = \cos x + i \sin x$ $e^{-ix} = \cos x - i \sin x$, and thence

$$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

[Prove $e^{-ix}(\cos x + i \sin x) = \text{constant} = 1$].

NOTE. The above results give general definitions of $\cos x$ and $\sin x$ for all values of $x$, imaginary as well as real. The exponential values of $\cos x$, $\sin x$ show that in all cases $\cos x^2 + \sin x^2 = 1$,

$$\sin(x+y) = \sin x \cos y + \cos x \sin y,$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y, \text{ etc.}$$

Also, $d\cos x = -\sin x\, dx$, $d.\sin x = \cos x\, dx$ $\cos o = 1$, $\sin o = o$.

76. If $x, y$ be functions of $\theta$, such that $dy = x d\theta$, $dx = -y d\theta$, and $x=1$, $y=o$ when $\theta=o$, show that $x = \cos\theta$, $y = \sin\theta$.

[ Prove $x \cos\theta + y \sin\theta = \text{constant} = 1$, and $x \sin\theta - y\cos\theta = \text{constant} = o$, and solve for $x, y$].

## 66. Expansions in Series

77. If $\tan x$ can be expanded in ascending powers of $x$, show that $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$

[Put $\tan x = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \ldots$ and determine the co-efficients by successive derivation and $x = o$].

78. If the following functions can be expanded in ascending powers of the variable given, then the expansions are:

(a) $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$

(b) $(1+x)^p$

$$= 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots$$

(c) $e^x \sec x = 1 + x + x^2 - \frac{2x^3}{3} + \ldots$

(d) $\log\cos x = -\left(\frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + ..\right) = \log\sin\left(\frac{\pi}{2} - x\right)$

Note that $\log x$, $\log\sin x$, $\cot x$, $\log\cot x$, etc., are functions that *cannot* be expanded in ascending powers of $x$ such as $A + Bx + Cx^2 + \ldots$

(e) Maclaurin's Theorem. $fx =$

$$fo + f'o.x + f''o.\frac{x^2}{2\,!} + f'''o.\frac{x^3}{3\,!} + \ldots + f^{(n-1)}o.\frac{x^{n-1}}{(n-1)\,!} + \ldots$$

where $f'x$, $f''x$, $f'''x$, ... $f^{(n-1)}x$, .. are the successive derivatives of $fx$.

Note. This expansion certainly fails when $fx$, $f'x$, $f''x$, $f'''x$, ..., are not all finite continuous functions for every value of $x$ between $o$ and its final value used in the expansion. When these derivations are all finite and continuous from $o$ to $x$, it can be shown (Article 68) that the difference between $fx$ and the first $n$ terms of the series is exactly $f^{(n)}x'.\frac{x^n}{n\,!}$, where $x'$ is some number between $o$ and $x$. Although $x'$ cannot be determined otherwise than that it is between $o$ and $x$, yet this form of the difference enables us to assign a superior value of it by using the largest value of $f^{(n)}x'$ as $x'$ changes from $o$ to $x$, and if this superior value approaches zero as $n$ approaches infinity, then certainly ($e$) must be true.

(f). Taylor's Theorem.

$$f(x+y) = fx + f'x.y + f''x.\frac{y^2}{2\,!} + \ldots$$

[The remainder after $n$ terms is $f^{(n)}x'.\frac{y^n}{n!}$. ($x'$ between $x$ and $x+y$)].

(g) $fx = fa + f'a\,(x-a) + f''a\,.\frac{(x-a)^2}{2!} + \ldots$

[Remainder after $n$ terms $= f^{(n)}x'.\frac{(x-a)^n}{n!}$, $x'$ between $a$ and $x$].

NOTE. From ($g$) show that if $a$ is a root of $fx = o$ then $x-a$ is a factor of $fx$ and conversely. Defining: *$a$ is an $n$-multiple root of $fx = o$ when $(x-a)^n$ is a factor of $fx$*, show that the double roots are common roots of $fx = o$, $f'x = o$, and that $x-a$ (if $a$ is a double root) is a common factor of $fx$, $f'x$. So, *triple roots* of $fx = o$ are roots of the greatest common factor of $fx$, $f'x$, $f''x$, and so on.

## 67. Maximum and Minimum Values

$fx$ being a finite continuous function of a real variable $x$, it is said to be *increasing* for a given value of $x$ when its value increases as $x$ *increases* from its given value and *decreases* if $x$ *decreases;* and it is said to be *decreasing* when its value *decreases* as $x$ *increases* from its given value, and *increases* if $x$ *decreases.* Also $fx$ is a *maximum* value when for immediately less values of $x$, $fx$ is increasing, and for immediately greater values of $x$, $fx$ is decreasing; while $fx$ is a *minimum value* if it is similarly changing from decreasing to increasing at the given value of $x$. The greatest value of $fx$ is necessarily a maximum value, and its least value a minimum value, but there may be other maximum and minimum values.

In other words, maximum and minimum values of $fx$ are only greatest and least values of $fx$ for values of $x$ in the immediate neighborhood of the given value, both greater and less. Since $\frac{dfx}{dx} = \lim \frac{\Delta fx}{\Delta x}$, therefore $\frac{dfx}{dx}$ must have the same sign as $\frac{\Delta Fx}{\Delta x}$ for small values of $\Delta x$, supposing that $\frac{dfx}{dx}$ *is neither zero nor discontinuous.* Hence:

(a) $fx$ is increasing when $f'x$ is positive.

(b) $fx$ is decreasing when $f'x$ is negative.

(c) $fx$ is a maximum when $f'x$ is changing from positive to negative, *i.e.*, when $f'x$ is positive for values of $x$ immediately less than its given value, and negative for values of $x$ immediately greater than its given value.

(d) $fx$ is a minimum when $f'x$ is changing from negative to positive.

(e) A maximum or minimum value of $fx$ can only occur for a value of $x$ that makes $f'x$ zero or discontinuous.

For, if $f'x$ is not zero or discontinuous, it will be positive or negative, which is case (a) or (b). The most common discontinuity is $f'x = \infty$.

Find the maximum and minimum values of the following functions:

78. $y = 5 + 8x - x^2$.

[$\frac{dy}{dx}=2\,(4-x)$, which changes from $+$ to $-$ at $x=4$, *i.e.*, it is positive when $x<4$ and negative when $x>4$. Therefore $y=5+32-16=21$ is the maximum value of $y$. Also $y$ can be as much less than zero as we please by taking $x$ large enough.]

79. $y=4+(x-3)^{\frac{2}{3}}-(x-3)^{\frac{5}{3}}$.

[$\frac{dy}{dx}=\frac{17-5x}{3(x-3)^{\frac{1}{3}}}$; which is zero for $x=17/5$, and discontinuous for $x=3$. At $x=17/5$, $\frac{dy}{dx}$ changes from $+$ to $-$ and $y$ is a minimum; at $x=3$, $\frac{dy}{dx}$ changes from $-$ to $+$ and $y$ is a maximum. Also $y$ can be as great as we please by taking $x$ enough less than zero, and $y$ can be as much less than zero as we please by taking $x$ great enough, so that both the maximum and the minimum values of $y$, are only with reference to adjacent values.]

80. $y=a\sin x+b\cos x$.

[$\frac{dy}{dx}=b\sin x-a\cos x$; $\tan x=a/b$, $y=\pm\sqrt{(a^2+b^2)}$.

$\frac{d}{dx}\frac{dy}{dx}=-(a\cos x+b\sin x)=-y$. Therefore, when $y$ is positive, $\frac{dy}{dx}$ is *decreasing*, by (*b*) with $\frac{dy}{dx}$ in place of $y$, and remembering that $\frac{dy}{dx}$ is zero for the given value of $x$, therefore it changes from $+$ to $-$ or

$y=\sqrt{a^2+b^2}$ is a maximum value. When $y$ is negative $\dfrac{dy}{dx}$ is *increasing*, and being zero, it is therefore changing from $-$ to $+$, and $y=-\sqrt{(a^2+b^2)}$ is a minimum value. These are true greatest and least values of $y$, since $y$ cannot increase or decrease indefinitely.]

81. $y=(x+1)^2\ (x-3)^3$.

[min., $x=-1$; max., $x=2$; min., $x=3$].

82. $w=x^2+y^2$, where $lx+my+n=o$.

[$w=n^2/(l^2+m^2)$, a minimum].

83. Find the largest rectanglar area that can be enclosed by a boundary of 200 feet. [2500 sq. feet.]

84. Find the largest rectangle that can be cut out of a circular sheet 6 feet in diameter. [18 sq. feet.]

85. Find the altitude of the maximum rectangle that can be cut from an isoscles triangle, one side being part of the base. [$\frac{1}{2}$ altitude triangle.]

86. Find the altitude of the maximum right cone that can be inscribed in a sphere of radius $a$.

[Let $a+x=$ altitude, $y=$ radius of base $=\sqrt{(a^2-x^2)}$; $x=\frac{1}{3}a$ for required maximum.]

87. Find the basin of largest volume, round or square, that can be made with a given number of square feet of tin. [Width = double the height].

88. A Norman window consists of a rectangle surmounted by a semicircle. Given the perimeter of the

frame, what dimensions give the window that will admit most light. [Height = width].

89. How far must one stand from the base of a column to obtain the largest angle of vision of a statue on the top.

[The distance is a mean proportional between the entire height of column and statue, and the height of the column].

## 68. Remainder in Maclaurin's Theorem.

Lemma. *If $Fz$, and its derivative $F'z$, be real and continuous functions of the real variable $z$ from $z=a$ to $z=b$, and if $Fa=Fb$, then will $F'z$ be zero for same value of $z$ between $a$ and $b$.*

For, when $z$ changes continuously from $a$ to $b$, $Fz$ must in the beginning either increase from the value $Fa$, or decrease from that value, and since it returns to the same value ($Fb=Fa$) in the end (and does so by continuous change of value), therefore there must be an intermediate value of $z$ at which $Fz$ changes from increasing to decreasing or from decreasing to increasing. Let $z=x'$ be such intermediate value of $z$; then by Art. 67, $Fx'$ is a maximum or a minimum value of $Fz$, and therefore $F'x'$ is either zero or discontinuous, and since the possibility of discontinuity is excluded by supposition, therefore $F'x'=o$.

Theorem. *The remainder in Maclaurin's theorem after $n$ terms is $f^{(n)}x' \cdot \frac{x^n}{n!}$, where $x'$ is some value between $o$ and $x$.*

For, let such remainder be $R\frac{x^n}{n!}$, so that $R$ is that function of $x$ which is given by the equation

(a) $fx=$

$$fo+f'o.x+f''o.\frac{x^2}{2!}+\dots+f^{(n-1)}o.\frac{x^{n-1}}{(n-1)!}+R\frac{x^n}{n!}$$

The conditions to which $fz$ and its derivitives $f'z$, $f''z$, $\dots f^{(n)}z$, must comply are that they are all real and continuous functions of the real variable $z$, from $z=o$ to $z=x$; and we can then therefore make up from these functions, the following function $Fz$ which, with its derivative $F'z$, are also real and continuous from $z=o$ to $z=x$:

$$Fz=fz+(x-z)f'z+\frac{(x-z)^2}{2!}f''z+\dots$$
$$+\frac{(x-z)^{n-1}}{(n-1)!}f^{(n-1)}z+\frac{(x-z)^n}{n!}R$$

$$F'z=f'z-f'z+(x-z)f''z-(x-z)f''z+\frac{(x-z)^2}{2!}f'''z-\dots$$
$$+\frac{(x-z)^{n-1}}{(n-1)!}(f^{(n)}z-R)$$
$$=\frac{(x-z)^{n-1}}{(n-1)!}(f^{(n)}z-R),$$

since preceding terms all cancel.

But $Fz=fx$ when $z=o$, in consequence of the value of $R$ given in (a); and $Fz=fx$ when $z=x$, in consequence of the vanishing of every power of $x-z$. Hence, by the lemma, $F'z=o$ for some value of $z$, between $o$

and $x$, say $z=x'$. Substituting this value of $z$ in $F'z$ and dividing out factors not zero, we find $R=f^{(n)}x'$.

90. Show that the error of $\log(1+x)=x-\frac{x^2}{2}+\ldots\mp\frac{x^n}{n}$ is between $x^{n+1}/(n+1)$ and $x^{n+1}/(n+1)(1+x)^{n+1}$, so that when $x$ is positive and not greater than 1, $\log(1+x)=x-\frac{x^2}{2}+\ldots$ How many terms of $\log 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ must be taken to compute $\log 2$ to an error between .0001 and .00005?

91. Show that the errors of

$$\sin x=x-\frac{x^3}{3!}+\ldots\mp\frac{x^{2n-1}}{(2n-)!},$$

$$\cos x=1-\frac{x^2}{2!}+\ldots\mp\frac{x^{2n-2}}{(2n-2)!},$$

are $\pm\sin\theta x.x^{2n}/(2n)!$, $\pm\cos\theta x.x^{2n-1}/(2n-1)!$ where $\theta$ is some number between $o$ and 1. Show that these errors are smaller than $\left\{\frac{x^2}{n}\right\}^n/n!$, $x\left\{\frac{x^2}{n}\right\}^{n-1}/n!$, respectively; and that therefore, however large $x$ may be, $n$ can be taken large enough so that the above approximatons to $\sin x$, $\cos x$ are as accurate as we please. How many terms of these series must be taken to compute $\cos 1$, $\sin 1$ to errors certainly smaller than .0000001?

92. If $fx=x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$, where $n$ is a positive integer, and $a_1$, $a_2$, ... $a_n$, are real numbers, show that between two real roots of $fx=o$, lies at least one real root of $f'x=o$.

## CHAPTER III

### Concrete Representation

**69.** Algebraic quantities are represented by concrete quantities such as length, area, volume, etc. Negative numbers are represented only by the assignment of opposite characters of measurement, and then a negative measurement of one character means the corresponding positive measurement of the opposite character. E.g., $-2$ units to the right $=2$ units to the left, $-3$ units up $=3$ units down, $-4$ radians counter clockwise $=4$ radians clockwise, etc. Imaginary numbers can be represented by *directed lengths in a plane* in accordance with the principle that $\sqrt{-1}$ denotes change of direction through a counter-clockwise right angle, as $\sqrt{-1}$ units to the right $=1$ unit up. This is applicable when the concrete quantities are such as forces acting at a point, but not for ordinary lengths, or areas or volumes.

**70.** The differential of a variable quantity must be a quantity of the same kind. In fact, the change of value, the proportional to this change of value, and consequently its limit the differential, must be the same kind of quantities as the given variable. In other words, the differential of a length is a length, of an area, an area, of a force, a force, etc. Concrete representation of variable numbers will therefore give corresponding representations of their differentials, and the determination of the differentials from the variables is important not only for its applications to concrete problems, but also because it gives concrete ideas of differentiation that illustrate this algebraic process and its principles.

**71.** Let $OX$, $OY$ (Fig. 1) be horizontal and vertical axes of reference in the plane of the paper. A variable point $P$ in this plane is determined by two variables $x$, $y$ called its co-ordinates, which are respectively the measures of the distances of $P$ to the right, and up, from the axes. Negative measures in these directions mean positive measures in the opposite directions. The first co-ordinate is called the *abscissa* of $P$, and is $OL=x$ units to the right (or briefly $OL=x$); the second co-ordinate is called the *ordinate* of $P$, and is $LP=y$ units up (or briefly $LP=y$). If $y$ be a definite real function of $x$, this means that each value of $x$ gives one and only one value of $y$, or that $P$ is represented on each vertical line by one and only one point; if $y=fx$ be a continuous function of $x$, then the locus of the point $P$ is a continuous curve, crossing each vertical line not more then once. As an example of discontinuity find the locus of $P$ from $x=1$ to $x=3$ when $y=x+$ integer part of $x$. Conversely, any continuous curve drawn from left to right, and crossing each vertical line once only, would, if we understand that $P$ always lies on this curve, make $y$ a definite function of $x$. In Figure 1 the curve drawn is actually a circle of center $C$, and vertical radius $AC$. The upper half of this circle corresponds to a different function of $x$ from the lower half. With a certain unit of length, we have $OB=8$, $BC=9$, $AC=5$. For $P=(x, y)$, any point on this circle, we find from the right triangle on $CP$ as hypothenuse, with sides parallel to the axes, that $(y-9)^2+(x-8)^2=25$, *i.e.*, $y=9\pm\sqrt{25-(x-8)^2}$ are the two functions in question.

**72.** The curve $y=fx$ is *smooth* when it has a definite tangent $PT$ at each point $P$, and when the direction of this tangent changes continuously for continuous variation of $P$. The tangent at $P$ is defined as *the limiting position of the indefinitely produced chord $PP'$ as $P'$ approaches coincidence with $P$.* This condition of smoothness is in fact the condition that $fx$ is differentiable and that such differential, $f'x\,dx$, is a continuous function. Continuity and smoothness are implied conditions on all curves. There may be exceptional or *singular points*, in this respect, but the continuous changes of value of the independent variable that are considered in general statements must not include such singular points.

**73.** In a given curve there are other functions of the abscissa $x$, of $P$, besides the ordinate $y$. Thus, let $A$ be an assigned *initial* position of $P$ on the curve, and let the tangent line $PT$ and the normal line $CP$ (perpendicular to the tangent) meet $OX$ in $M$, $N$, and also meet a perpendicular to $OP$ through $O$ in $M'$, $N'$. Then:

the *arc* of the curve is the arc $AP=s$;

the *ordinate area* is the area $ABLP=u$ (described by the ordinate);

the *slope angle*, is the angle $XMP=\phi$ radians;

the *slope* is $\tan\phi$;

the *tangent* and *normal* (lengths) are, $MP$ and $PN$;

the *subtangent* and *subnormal* are, $ML$ and $LN$;

the *polar radius* and *angle* are $OP = r$ and $\angle XOP = \theta$ radians;
the *polar radius area* is the area $OAP = v$;
the *polar slope angle* and *slope*, are $\angle OPM = \psi$ radians, and $\tan \psi$
the *polar tangent* and *normal*, are $M'P$ and $PN'$
the *polar subtangent* and *subnormal*, are $M'O$ and $ON'$.

**74.** The co-ordinates $r$, $\theta$ are *polar* co-ordinates of $P$; the unit of measure for the angle $\theta$ is a *counter-clockwise* radian; the unit of measure for $r$ is the *unit to the right* turned through the angle $\theta$, so that it is in *the direction* $\theta$; $r$ is therefore positive or negative according as the direction $\theta$ is towards or from $P$. We generally suppose $\theta$ taken so that $r$ is positive. The unit of measure on $M'N'$ is in the direction $\theta - \frac{\pi}{2}$. The units on tangent and normal are in the direction $\phi$ and $\phi + \frac{\pi}{2}$; the direction $\phi$ can be taken as the direction of *increase* of $s$. The area described by the ordinate $y$ is divided into positive and negative parts determined by the product of the sign of the value of $y$ and its positive or negative direction of motion along $OX$. The area described by the radius vector $r$ is also divided into positive and negative parts according to its positive (counter-clockwise) or negative direction of turning about $O$, whether $r$ is positive or negative. Conventions of sign are for definiteness of general statements, *i.e.*, with such conventions, general theorems can be made holding for any position of $P$ on its given locus, that must otherwise be separated into several distinct theorems depending upon the position of $P$. In other words, results that are obtained from a construction in which all the quantities are positive will hold for any possible construction when proper conventions of sign are used to interpret the quantities, which would not thus generally hold when magnitude only is concerned. This is because conventions of sign make continuously varying quantities change from positive to negative when their magnitudes are to change from additive to subtractive, as the position of $P$ changes continuously.

**75.** To construct the differentials of abscissa $x$, ordinate $y$, and arc $s$, of a given curve, for assigned values of $x$, $dx$.

*Let $P$ be the point on the curve whose abscissa is $x$; take $PR = dx$ units to the right; draw the tangent at $P$, and draw $RS$ parallel to $OY$ to meet this tangent in $S$; then is $RS = dy$ units up, and $PS = ds$ units in the direction of increase of $s$.*

For, let $P'$ be the point on the given curve whose abscissa is the new value $x'$; let the new ordinate $y' = L'P'$ meet the line $PR = dx$ at $Q$; lay off on $PR$ the length $PR' = N.PQ$; draw $R'S'$ parallel to $OY$ to meet the chord $PP'$ in $S'$. Then by similar triangles, $R'S' = N.QP'$, and since $PQ = \Delta x$, $QP' = \Delta y$, therefore, $PR' = N.\Delta x$, $R'S' = N.\Delta y$. The differential process

$$\lim N\Delta x = dx, \text{ or } \lim PR' = PR,$$

consists in making $Q$, and therefore $P'$, approach coincidence with $P$, while making $N$ correspondingly increase so that the point $R'$ approaches coincidence with $R$. Two constructions are shown in the figure, the first is lettered as described and $N=3$, the second is unlettered, $Q$ is nearer to $P$ than in the first construction, $R'$ nearer to $R$, and $N=7$. We are to imagine a series of such constructions, unlimited in number, in which $Q$ is taken nearer and nearer to $P$, with the object of determining the limit of $S'$ knowing that the limit of $R'$ is $R$. It is easily seen that $S'$ approaches $S$, for, in the first place, $R'S'$ is by construction always parallel to $OY$, and therefore its limiting position is a line $RS$ parallel to $OY$, and secondly, since $P'$ approaches $P$, and $S'$ lies by construction on the chord $PP'$ (produced), therefore $S'$ must approach coincidence with a point on the tangent at $P$, which is by definition the limiting position of the chord $PP'$ produced. Hence $dy = \lim N\Delta y = \lim R'S' = RS$, where $RS$ is a line parallel to $OY$, and meeting the tangent $PT$ in $S$.

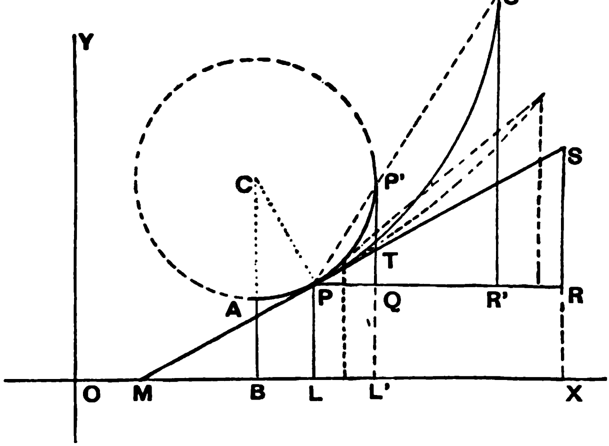
Next produce the chords from $P$ to each point of the arc $PP' = \Delta s$, in the ratio $N:1$, and let arc $PS'$ be the curve in which such extended chords terminate. The arc $PS'$ is then similar to *arc* $PP'$ by construction, and its length is $N.\,arc\,PP' = N.\,\Delta s$; both arcs have also the same tangent $PT$, since the tangent is determined by the limiting position of the same chords produced, in either case. Thus when $P'$ is so near to $P$ that the arc $PP'$ is always between its chord and tangent and of one direction of bending throughout, the similar arc $PS'$ must lie between its chord $PS'$ and tangent $PS$, and be of one direction of bending throughout. Hence as $P'$

approaches $P$, the arc $PS'$ must approach point to point coincidence throughout with the straight line $PS$, since $PS'$ does so; *i.e.*,

$$ds = \lim N\Delta s = \lim arc\ PS' = PS.$$

Observe that the two triangular figures $PQP'$, $PR'S'$, each with an arc side, are similar figures, with $P$ as center of symmetry, and $N$ as ratio of similitude. Since $PR'S'$ approaches coincidence with the triangle $PRS$, it

FIG. 1.




appears that the difference figure $PQP'$ approaches similarity with the differential triangle $PRS$, as its sides indefinitely diminish. The difference figure reduces to the point $P$; we have left, however, in the triangle $PRS$ what might be called its ultimate form.

(a) *The limiting ratio of any arc to its chord as the arc is taken smaller and smaller, is unity.*

For, $PP' : \text{arc}\, PP' = PS' : \text{arc}\, PS'$, whose limit is $PS : PS = 1$

(b) *The slope of the curve* $y = fx$ *at the point* $(x, y)$ *is* $dy / dx = f'x$.

For, the slope is $\tan \phi = RS / PR = dy / dx$.

(c) Inverse Principle 1. *If* $dw = o$ *identically, then* $w = constant$.

[Or, *if* $du = dv$ *identically, then* $d(u - v) = o$ *identically, and* $u - v = constant$.]

For the slope of $y = fx$ being $dy / dx$, then if $dy = o$ for a particular value of $x$, at such point $(x, y)$ $S$ coincides with $R$, or the tangent is parallel to $OX$; if $dy = o$ for every value of $x$, then the tangent is parallel to $OX$ at every point, and the curve $y = fx$ must be a straight line parallel to $OX$, so that $y = fx$ remains constant as $x$ changes. This proves the principle for real functions of one real variable. More generally, if $w$ be a real function of real independent variables $x$, $y$, then $dw, = \dfrac{\partial w}{\partial x} dx + \dfrac{\partial w}{\partial y} dy$, can be identically zero only when $\dfrac{\partial w}{\partial x} = o, \dfrac{\partial w}{\partial y} = o$, identically, since $dx$, $dy$, are arbitrary values. Thus $w = f(x, y)$ is constant when either $x$ or $y$ changes alone, whatever value the other variable may have, and it must then be constant when both vary, since $\Delta f(x, y) = \Delta_x f(x, y) + \Delta_y f(x', y) = o + o = o$. Similarly for real functions of any number of real variables, and this includes imaginary variables, regarded as depending upon their real components. If $w$ is imaginary, it is $w = w_1 + w_2 \sqrt{-1}$, where $w_1$, $w_2$

are real, and $dw = dw_1 + dw_2 . \sqrt{-1} = o$ only when $dw_1 = o$, $dw_2 = o$, so that $w_1$, $w_2$, and therefore $w$, will be constant, if $dw = o$, identically.

**76.** To construct the differentials of the polar radius and angle, $r$, $\theta$:

*Draw from $P = (r, \theta)$, the length $PS = ds$ on the tangent at $P$,* (Art. 75)*; draw $SR_1$ perpendicular to the polar radius $OP$ at $R_1$; with center $O$ draw the arc $PS_1$ equal to $R_1 S$ in length; then is*

$$PR_1 = dr, \quad R_1 S = r d\theta, \quad \angle POS_1 = d\theta.$$

For take on the given curve, the point $P' = (r', \theta')$, and with $O$ as center draw the arc $P'Q'$ to meet $OP$ in $Q'$; produce $PP'$ into $PS' = N.PP'$, and the chords from $P$ to the arc $PP'$ in the same ratio, so as to determine the arc $PS'$ similar to the arc $PP'$; draw $S'O'$ parallel to $P'O$ to meet $PO$ in $O'$, and with $O'$ as center draw the arc $S'R''$ to meet $OP$ in $R''$; then by construction, and similar figures, $\angle POP' = \Delta\theta$, arc $Q'P' = r'\Delta\theta$, $PQ' = \Delta r$, arc $PP' = \Delta s$, and arc $R''S' = N$ arc $Q'P' = r'N\Delta\theta$, $PR'' = NPQ' = N\Delta r$, arc $PS' = N\Delta s$. When $P'$ approaches $P$, the arc and chord $PS'$ approach the common limit $PS = ds$ on the tangent at $P$ (Art. 75); also $S'O'$, which is parallel to $P'O$, approaches a parallel to $PO$ through $S$, and consequently the arc $R''S'$, which meets its radii perpendicularly, approaches coincidence with the line $R_1 S$ that is pependicular to $OP$ at $R_1$. Hence:

$$R_1 S = \lim \text{arc}\, R''S' = \lim r' N \Delta\theta = r\, d\theta$$
$$PR_1 = \lim PR'' = \lim N\Delta r = dr$$
$$\angle POS_1 = \text{arc}\, PS_1 / OP = R_1 S / OP = d\theta$$

Observe that $PR_1S$ is the ultimate form of the difference figure $PQ'P'$, since it is the limit of the similar figure $PR''S'$.

**77.** *The differential of the ordinate area* $u=ABLP$ (Fig. 1), *is the area* ($ydx$) *of the rectangle on* $LP$ *and* $PR$.

For $\Delta u=$ area $PLL'P'=y_1\Delta x$, where $y_1$ is some ordinate between $LP$ and $L'P'$, so that when $P'$ approaches $P$, we shall have $\lim y_1=y$; and hence

(a) $du=\lim N\Delta u=\lim y_1 N\Delta x=y\,dx.$

**78.** *The differential of the polar radius area,* $v=OAP$, (Fig. 2) *is the area of the triangle* $OPS$ *or of the sector* $OPS_1$.

For $\Delta v=$area $OPP'=$triangle $OPP'$ plus segment $PP'$ or $\Delta v=$sector $OQ'P'$ minus figure $PQ'P'$. Also since similar areas of Fig. 2 are as $N^2:1$, therefore

$N.$ segment $PP'=$ segment $PS'/N,=o$ when $N=\infty$,
$N.$ figure $PQ'P'=$ figure $PR''S'/N,=o$ when $N=\infty$.

Hence $dv=\lim N.$ triangle $OPP'=\lim$ triangle $OPS'=$ triangle $OPS$, or $dv=\lim N$ sector $OQ'P'=$sector $OPS$, since $\lim OQ'=OP$, $\lim N$ arc $Q'P'=$ arc $PS_1$.

(a) $dv=\frac{1}{2}r^2\,d\theta=\frac{1}{2}(x\,dy-y\,dx)$

For drawing the ordinates $LP$ and $KS$,

$$\begin{aligned}dv&=\text{area } OPS=\tfrac{1}{2}OP.R_1S=\tfrac{1}{2}r^2\,d\theta\\&=\tfrac{1}{2}\{OK.KS-(LP+KS)LK-OL.LP\}\\&=\tfrac{1}{2}\{(x+dx)(y+dy)-(2y+dy)\,dx-xy\}\\&=\tfrac{1}{2}(x\,dy-y\,dx).\end{aligned}$$

Also, from Fig. 2,

(b) $r^2 = x^2 + y^2$, $\theta = \tan^{-1}(y/x)$,
which give by differentiation,

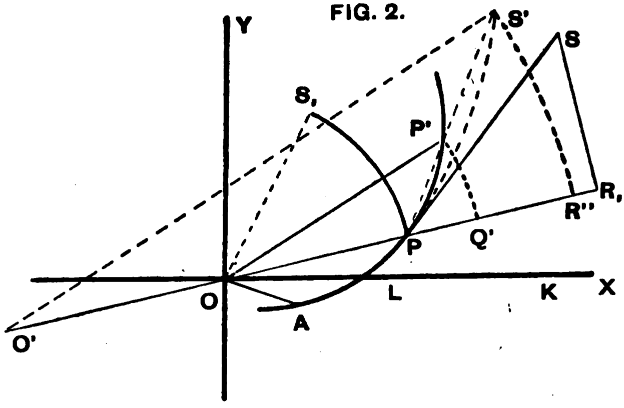
(c) $r\,dr = x\,dx + y\,dy$, $r^2 d\theta = x\,dy - y\,dx$.

From the right triangles $PRS$, $PR_1S$, we find

(d) $ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$.

Also verify the second equation of (d) by squaring and adding equations (c).

FIG. 2.




**79.** Observe that as shown in Art. 20, the preceding differentials all vary proportionally with $dx$, for an assigned value of $x$, and each can therefore be expressed in the form $fx.dx$. For example, let the given curve be $y = x^2$; then, $dy = 2x\,dx$; $ds = \sqrt{(dx^2 + dy^2)} = \sqrt{(1+4x^2)}\,dx$; $du = y\,dx = x^2 dx$; $dv = \frac{1}{2}(x\,dy - y\,dx) = \frac{1}{2}x^2 dx$, etc. Inverse Prin. 1 will then determine $s$, $u$, $v$ by anti-differentiation as soon as the initial point $A$ is assigned; if this be $x=o$, then

$$s=\tfrac{1}{2}x\sqrt{(1+4x^2)}+\tfrac{1}{4}\log(2x+\sqrt{1+4x^2}),$$
$$u=x^3/3=xy/3;\ v=\tfrac{1}{2}u.$$

**80.** DEFINITION. *The state of change* of a variable quantity *at* given values of its variables, is that state in which it would change by the value of its differential when its variables are changed by the values of their differentials.

E.g., $x^2$, at a given value of $x$, is in a state of change in which it would change by $2x\,dx$ where $x$ is changed by $dx$; at $x=3$, $x^2=9$, and would become, as it is then changing, 15 when $x=4$, 21 when $x=5$, and so on; at $x=5$, $x^2=25$, and would become 35 when $x=6$, 45 when $x=7$, and so on. Again, a particle which falls $16\,t^2$ feet in $t$ seconds, would, as it is falling at time $t$, fall $32tdt$ feet in $dt$ seconds, or $32t$ feet per second. So, when we say "that train is running 20 miles an hour" we mean to express its state of motion at the time of observation, not how far it will run in an hour.

**81.** The fundamental properties of the state of change of a variable quantity are:

(a) *It is a uniform state; i.e.*, the changes of the quantity vary proportionally with the changes of the variables (Art. 20), or more generally the sum of corresponding changes of the quantity and its variables, are also corresponding changes of the same, (proven by

$$dw=\frac{\partial w}{\partial x}dx+\frac{\partial w}{\partial y}dy).$$

(b) *The change of value of a quantity in its state of change, and its actual change of value, may be made as nearly equal as we please by taking the changes of the variables small enough*—approximate equality between very small quantities being taken in the sense that proportionals to them of sensible magnitude are approximately equal, otherwise *any two* very small quantities would be approximately equal, and the stated property would be no property. For, let $w$ be the quantity, and $x$, $y$, its variables; then by definition of differentiation, any changes $\Delta x$, $\Delta y$ can be taken so small that for a large multiplier $N$, and the assigned values $dx = N\Delta x$, $dy = N\Delta y$, to which corresponds $dw = k$, say, we shall have $N\Delta w$ and $k$ as nearly equal as we please. Hence assigning anew, $dx = \Delta x$, $dy = \Delta y$, to which corresponds $dw = k/N$ (Art. 20), we shall have $dw$ and $\Delta w$ as nearly equal as we please in the sense that their proportionals $k$ and $N\Delta w$ of sinsible magnitude are so.

We may say that the differential $dw$, of a function of $x$, $y$, corresponding to sufficiently small values, $dx = \Delta x$, $dy = \Delta y$, is *the principal part* of $\Delta w$.

E.g., taking $dx = \Delta x = PQ$ (Fig. 1) sufficiently small, then $dy = QT$, $ds = PT$, $du =$ rectangle $LQ$, $dv =$ triangle $OPT$, are the principal parts of $\Delta y = QP'$, $\Delta s =$ arc $PP'$, $\Delta u =$ area $PLL'P'$, $\Delta v =$ area $OPP'$.

**82.** When a point $P$ is moving upon a fixed curve (Fig. 1), the quantities $x$, $y$, $s$, $r$, $\theta$, $u$, $v$, etc., corresponding to the given position of $P$ have states of change characterized by the corresponding changes $dx$, $dy$, $ds$, $dr$, $d\theta$, $du$, $dv$, etc. These vary proportionally together so that the ratio of any two is *the change of the first quantity per unit of change of the second.* Thus $y$ would change by $dy$, and $s$ by $ds$, when $x$ changes by $dx$, and $dy/dx$, $ds/dx$ are the rates of change of $y$, $s$, as to $x$. If $P$ has its given position at a corresponding time $t$, then $x$, $y$, $s$, etc., are functions of $t$, and $dx$, $dy$, $ds$, etc., are differentials corresponding to any assigned length of time $dt$. Considering the motion of $P$ with respect to change of distance, it would move in time $dt$ a distance $ds$ on the tangent at $P$, so that $ds/dt$ is the *speed* of $P$ at time $t$, and it is tangential in direction. Using a dot over a quantity to denote its derivation as to the time, then $\dot{s}=ds/dt$ is a quantity depending upon $t$, and its change per unit of time is $\ddot{s}=d\dot{s}/dt$, called *the tangential acceleration.* Considering $P$ as a small particle moving in its path in consequence of force acting upon it, as in the case of a thrown pebble moving under the action of gravity and the resistance of the air, the tangential acceleration is not the whole acceleration, but only that component of the whole acceleration that is in the direction ($\phi$) of motion. The whole acceleration is by Newton's laws of motion, the time derivative of the velocity, of which the speed is simply the magnitude; in other words, variation of direction as well as of magnitude, must be taken into account in the differentiation. (See Art. 85).

## Curvature

**83.** The curvature of a curve at any point is its *change of direction (in radians) per unit length of arc; i.e., the* *curvature is $d\phi / ds$ where $\phi$ is the slope angle (in radians) and $s$ is the arc length.*

(a) *When the curvature is zero at every point the curve is a straight line.* [*If $d\phi = o$ identically, then $\phi =$ constant*].

(b) *The curvature of a circle is the same at every point and equal to the reciprocal of its radius.*

Let $C$ be the center of a circle of radius $a$, $A$ its lowest point, and $P$ any other point (Fig. 1); then $\phi = \angle XMP = \angle ACP = \text{arc } AP / CP = s/a$; thus $s = a\phi$, and $ds = a\,d\phi$, or $d\phi/ds = 1/a$.

**84.** *The circle of curvature at a point $P$ on a given curve*, is the tangent circle at that point with the same direction and magnitude of curvature as the curve. Its radius is therefore $R = ds/d\phi$, and its center $C = (\mathbf{x}, \mathbf{y})$, is distance $R$ from $P$ in the direction $\phi + \pi/2$. If $R$ is negative this means that the center is actually in the opposite direction, since $ds/d\phi$ will be positive or negative, according as $\phi$ increases or decreases as $s$ increases, *i.e.*, according as the curve bends towards the direction $\phi + \pi/2$ or $\phi - \pi/2$. Thus equating projections of $OC$ and the broken line $OPC$ upon the axes, we find

(a) $\mathbf{x} = x + R\cos(\phi + \pi/2) = x - R\sin\phi = x - dy/d\phi$.
(b) $\mathbf{y} = y + R\sin(\phi + \pi/2) = x + R\cos\phi = x + dx/d\phi$.

In rectangular co-ordinates,

$$ds = \sqrt{(dx^2 + dy^2)} = \sqrt{(1 + p^2)}\, dx, \qquad [p = dy/dx = \tan\phi].$$

$$d\phi = dp/(1 + p^2); \; R = (1 + p^2)^{\frac{3}{2}} / (dp/dx).$$

In polar co-ordinates,

$$ds = \sqrt{(r^2\,d\theta^2 + dr^2)} = \sqrt{(r^2 + q^2)}\,d\theta,$$

$$[q = dr/d\theta = r\cot\psi];$$

$$\phi = \psi + \theta, \qquad \text{[triangle } OMP\text{, Fig. 1]};$$

$$d\phi = d\psi + d\theta = \{(r^2 + 2q^2)\,d\theta - r\,dq\}/(r^2 + q^2);$$

$$R = (r^2 + q^2)^{\frac{3}{2}}/(r^2 + 2q^2 - r\,dq/d\theta).$$

**Differentiation of Directed Quantities**

**85.** *The differential of a directed quantity OP* (Fig. 2) *that varies definitely with the time, and whose values add by the parallelogram law, is a directed quantity PS, whose component* $PR_1$ *along OP is the differential of the magnitude of OP, and the perpendicular component* $R_1S$ *is the product of the magnitude of P and its differential change of direction (in radians).*

According to the parallelogram law, (true for velocities, forces, etc.) $OP + PP' = OP'$, so that $\Delta . OP = PP'$, $N\Delta . OP = NPP' = PS'$, and when $P'$ approaches $P$, $d . OP = \lim PS' = PS$, a tangent to $s = \text{arc}\,AP$, of length $ds$. Also, the components of $PS$ along and perpendicular to $OP$, are $PR_1 = dr$, $R_1S = rd\theta$.

**86.** The velocity of $P$ is the time derivative of the displacement $OP$. Its magnitude is $\dot{s}$, and its direction is $\phi$. The acceleration of $P$ is the time derivative of the velocity. Its components are therefore $\ddot{s}$ in the direction $\phi$, (tangential) and $\dot{s}\,d\phi/dt = \dot{s}^2\,d\phi/ds = \dot{s}^2/R$, in the direction $\phi + \pi/2$ (normal), where $R$ is the radius of curvature of the path at $P$, (Arts. 85, 83). If we draw $O\dot{P}$ in the direction $\phi$, and in length

$\dot{s}=ds/dt$, then the path of $\dot{P}$ as $t$ varies is called *the hodograph of the motion of P;* and it appears that the acceleration of $P$ is in direction and magnitude, the velocity of the corresponding point $\dot{P}$ in its hodograph. If there is no force acting upon the particle $P$ at any time, then $\ddot{s}=o$ or $\dot{s}=$constant, and $d\phi/ds=o$, or the path is a straight line. No force acting means then uniform motion in a straight line. If $\ddot{s}=o$ and $d\phi/ds=$constant, then the particle is moving uniformly in a circle, and there is an acceleration towards the center at every point, of constant magnitude, $(\text{speed})^2/\text{radius}$.

In a particle constrained to move in a circle, this is the acceleration of the tension along the radius. It appears that the normal component of the force on a particle is the deflecting component, and the tangential component is the speed accelerator. In straight line motion, the normal component is zero, since the curvature is zero.

## Examples III.

1. In the parabola $ay=x^2$ find $P$, and construct $PRS$ of Fig. 1, in the cases: $x=2a$, $dx=a$; $x=a$, $dx=2a$; $x=o$, $dx=3a$; $x=-a$, $dx=-3a$; $x=-2a$, $dx=a$. With a given value of $x$, what change is made in $PRS$ by changing $dx$?

2. Construct points ($P$) and tangents ($PS$) of the semi-cubical parabola $ay^2=x^3$ for $x=o, a, 4a$. Why is there no point corresponding to a negative value of $x$?

3. Find one arc of the semi-cubical parabola from $x=o$ to any value of $x$.

$$\text{Ans.}\quad s=\frac{8a}{27}\left\{\left(1+\frac{9x}{4a}\right)^{\frac{3}{2}}-1\right\}$$

4. Find $P$ and construct $PR_1S$ of Fig. 2 for the equi-angular spiral $r=ae^{\theta}$, when $\theta=o$, $d\theta=a$; $\theta=\pi/6$, $d\theta=2a$; $\theta=\pi/3$, $d\theta=a$; $\theta=\pi/2$, $d\theta=-a$. Show that the radius always meets the curve at an angle of 45°.

5. Draw the points and tangents at $\theta=o$, $\pi/4$, $\pi/2$, $3\pi/4$, $\pi$, for the curves $r=2a\cos\theta$, $r=a\cos 2\theta$.

6. Find the curve in which $r=a$ when $\theta=o$, and whose polar radius meets the curve at a constant angle

$$\psi=\tan^{-1}c.\quad [r\,d\theta/dr=c;\ r=ae^{\theta/c}].$$

7. Find the length of the arc of the equi-angular spiral of Ex. 6, from $\theta=o$.

$$[ds=\sqrt{(1+c^2)}\,dr;\ s=\sqrt{(1+c^2)}\,(r-a)].$$

8. A point $P$ moves so that its distance ($x$) from a fixed *directrix* $OY$ is in a constant ratio ($e:1$) to its distance ($r$) from a fixed *focus* $F$; show that a tangent to the locus between the directrix and point of contact subtends a right angle at the focus.

[$r=ex$, $dr=e\,dx$; take $PS=ds$, where $S$ is on the directrix; then $dx=-x$, and $dr=-ex=-r=PF$, so that (Art. 76) $F=R_1$, the foot of the perpendicular from $S$ on $FP$.]

9. A point $P$ moves so that the sum of its distances

$(r, r')$ from the fixed focii $F$, $F'$, is a constant $(2a)$. Show that the tangent at $P$ bisects the angle between one focal radius and the other produced.

$r+r'=2a$, $dr+dr'=o$; therefore take $PR_1$ on $FP$ produced for $dr$, whence an equal length $PR'_1$ on $PF'$ is $dr'$, and the perpendiculars to $r$, $r'$ at $R_1$, $R'_1$ must meet in a point $S$ of the tangent at $P$ (Art. 76).

10. If the point $P$ moves so that the difference of the focal radii $r$, $r'$ of Ex. 9 is a constant, show that the tangent bisects the angle between the focal radii.

11. If the focal radii of $P$ as to fixed focii $F$, $F'$ satisfy the condition $r+2r'=3a$, find a construction for the tangent at $P$; similarly if $r-2r'=3a$.

12. Find the ordinate areas, and the polar radius areas from $O$, in the curves $ay=x^2$, $ay^2=x^3$, $a^2y=x^3$.

13. Find the ordinate area of the circle $x^2+y^2=a^2$ from $x=o$, and of the ellipse $x^2/a^2+y^2/b^2=1$.

Ans. $\frac{1}{2}x\sqrt{(a^2-x^2)}+\frac{1}{2}a^2\sin^{-1}\frac{x}{a}$, and $b/a$ times the same. The areas of circle $a$, and ellipse $(a, b)$ are found by putting $x=a$ and multiplying by 4, giving $\pi a^2$, $\pi ab$.

14. Find the polar radius area of the circle $r=a$ from $\theta=o$.

15. Find the polar radius area from $\theta=o$ of the curves of Exs. 1–6.

17. Find the volume of a hemisphere of radius $a$,

between its base and a parallel section at distance $x$; also the convex surface.

[The radius of the section at distance $x$ is $y=\sqrt{(a^2-x^2)}$, and if $V$ be the required volume $\Delta V=\pi y_1^2 \Delta x$ where $\pi y_1^2$ is the area of some section between the distances $x$ and $x+\Delta x$, so that $\lim y_1=y$. Thus $dV=\pi y^2 dx=\pi(a^2-x^2)dx$, $V=\pi x(a^2-\frac{1}{3}x^2)$. If $S$ be the convex surface, $s$ its arc section by a diametric plane perpendicular to the base, then $dS=2\pi y\,ds=2\pi a dx$ since $s$ is an arc of the circle $y=\sqrt{(a^2-x^2)}$ from $x=o$; and $S=2\pi ax$].

18. Find the moment of the spherical segment of Ex. 17 as to its base, and its center of volume.

[The moment of volume as to a plane is, *volume times distance from plane to center of volume*—distances on opposite sides of the plane having opposite signs. The moment of a volume equals the sum of the moments of its parts. These suffice to determine moment and center. Thus, $\text{mom}\,V'=\text{mom}\,V+\text{mom}\,\Delta V$, and hence $\Delta.\text{mom}\,V=\text{mom}\,\Delta V=x_1\Delta V$ where $x_1$ is some distance (to the unknown center of volume) between $x$, $x'$, so that $\lim x_1=x$, and $d.\text{mom}\,V=x\,dV=\text{mom}\,dV$, considered as concentrated at the distance $x$ to which it pertains. Hence $d\,\text{mom}\,V=\pi(a^2x-x^3)dx$, $\text{mom}\,V=\frac{\pi}{4}(2a^2x^2-x^4)$, and distance of center of volume $=(\text{mom}\,V)/V=\frac{3x}{4}\cdot\frac{2a^2-x^2}{3a^2-x^2}$. Take $x=a$ to make $V=$ vol. hemisphere, etc.]

19. Find the volume of a right circular cylinder of radius $x$ and altitude $c$ and its moment of inertia about its axis.

[$\Delta V=2\pi x_1 \Delta x . c$, where $x_1$ is between $x$ and $x'$. $dV=2\pi cx dx$, $V=\pi cx^2$. The moment of inertia of a volume as to an axis is, *the volume into the square of its radius of gyration as to the axis.* Such radius is between the longest and shortest radius to volume. Also the moment of inertia of a volume is the sum of the moments of inertia of its parts. Thus $\Delta$. mom-iner. $V=$ mom-iner. $\Delta V=x_1^2 \Delta V$, $d$. mom-iner. $V=x^2 d V=$ mom-iner. $d V$, considered as concentrated at distance $x$. Moment inertia $V=\pi c x^4/2$; radius of gyration $=x/\surd 2$].

20. Find the volume of a cone of altitude $x$ whose base area is $a^2$ when $x=1$; also find its moment as to a plane through its vertex perpendicular to its altitude, and the distance of its center of volume from the plane.

21. Find the moment of inertia, radius of gyration about its axis, and the convex surface, of a cone of revolution, of altitude $x$ and semi-vertical angle $\beta$.

[$\pi x^5 \tan\beta^4/10, x\tan\beta \surd. 3$, $\pi x^2 \sec\beta \tan\beta$].

22. Show that when $x=4$ the quantity $\surd x$ is changing one-fourth as fast as $x$, and that for small values of $h$, $2+\frac{1}{4}h$ is the principal part of $\surd(4+h)$.

23. A man walks 3 feet per second towards a tower 80 feet high. If he should continue to approach the top as at 60 feet from the base, in what time would he reach the top?

[$s^2=6400+x^2$, $dx=-3dt$; $dt$ is required when $x=60$ and $ds=-100$, and is $55.5+..$ seconds].

24. Two men starting together walk in paths at right angles, each 3 feet per second; show that one leaves the other $3\sqrt{2}$ feet per second.

25. A vessel is anchored in 18 feet of water, and the cable passes through a sheave in the bow 6 feet above water. If the cable is hauled in 18 feet per minute, what is the speed and acceleration of the vessel when 30 feet of cable are out?

[If $l$ = cable out at start, $a$ = horizontal distance to anchor, and $s, x$ = same after $t$ minutes, then $(l-s)^2=(a-x)^2+\overline{24}^2$; $ds=18dt$, and when $l-s=30$, we have $\dot{x}=30$, $\ddot{x}=32$].

26. A particle $P$ moves in a plane curve about a fixed point $O$; find its radial and radial normal components of velocity and acceleration.

Take an initial axis $OX$, and let $\{r, \theta\}$ stand for a directed quantity whose magnitude is $r$ in the direction $\theta$ radians from $OX$, this symbol in particular standing for $OP$ so that $\theta=\angle XOP$, $r$ = length $OP$. Then by Art. 85, velocity

$$=d(OP)/dt=d\{r,\theta\}/dt=\{\dot{r},\theta\}+\{r\dot{\theta},\theta+\tfrac{\pi}{2}\}=O\dot{P}$$

To differentiate again, we have,

$$d\{\dot{r},\theta\}/dt=\{\ddot{r},\theta\}+\{\dot{r}\dot{\theta},\theta+\tfrac{\pi}{2}\}$$

$$d\{r\dot{\theta},\theta+\tfrac{\pi}{2}\}=\{\dot{r}\dot{\theta}+r\ddot{\theta},\theta+\tfrac{\pi}{2}\}+\{r\dot{\theta}^2,\theta+\pi\}$$

and adding, we find acceleration

$$=d(O\dot{P})/dt=\{\ddot{r}-\dot{r}\theta^2, \theta\}+\{2\dot{r}\dot{\theta}+r\ddot{\theta}, \theta+\frac{\pi}{2}\}$$

The required components are therefore,

*velocity*, $\dot{r}=dr/dt;$

$$r\dot{\theta}=rd\theta/dt$$

*acceleration*, $\ddot{r}-r\dot{\theta}^2=\frac{d}{dt}\frac{dr}{dt}-r\frac{d\theta^2}{dt^2};$

$$2\dot{r}\dot{\theta}+r\ddot{\theta}=\frac{1}{r}\frac{d}{dt}\left[r^2\frac{d\theta}{dt}\right]$$

27. If a particle $P$ move about $O$ in a plane curve so that the radius $OP$ describes equal areas in equal times then the whole acceleration is radial, and conversely. [If $v=ct$, then (Art. 78a) $dv=r^2\,d\theta=c\,dt$, or $r^2\,d\theta/dt=c$, a constant, and the radial normal acceleration is zero; so conversely. The radial acceleration is then the whole acceleration and is

$$f=\frac{d}{dt}\frac{dr}{dt}-r\frac{d\theta^2}{dt^2}=\frac{c^2}{4r^2\,d\theta}d\left[\frac{1}{r^2}\frac{dr}{d\theta}\right]-\frac{c^2}{4r^3}$$

$$=-\frac{c^2}{4r^2}\left[\frac{d}{d\theta}\frac{d}{d\theta}\frac{1}{r}+\frac{1}{r}\right],$$

from $1/dt=c/r^2\,d\theta$. This finds $f$ as soon as the path is known.]

28. If a planet $P$ move in an ellipse with the sun $O$ as focus, and the radius $OP$ describe equal areas in equal times, show that the force moving the planet is towards the sun and varies directly as its mass and inversely as the square of its distance.

[This was deduced by Newton, the premises being Kepler's laws obtained by astronomical observations, and it led to the law of gravitation. In the ellipse of focus $O$, major radius $a$, minor radius $b$, excentricity $e$, and direction $OX$ along the major axis toward the center, $1/r=a(1-e\cos\theta)/b^2$, and $f=-ac^2/4b^2r^2$. The constant $ac^2/4b^2$ is the force per unit mass per unit distance, since acceleration = force per unit mass.

29. If the cubes of the major radii of the orbits of any two planets are as the squares of their periodic times (Kepler's third law), show that the gravitational constant is the same for all planets.

[The period of one revolution being $T$, then *area of orbit* $=cT=\pi ab$, and the gravitational constant is $ac^2/4b^2=\pi^2a^3/4T^2$].

30. Find the differential equations of the curve formed by a flexible cable with fixed ends and supporting a load continuously distributed along the cable.

[Take a tangent and vertical line through the lowest point $O$ of the curve (so that the tangent is horizontal) for axes of reference; let $P=(x, y)$ be any point of the cable; $H$ the tension at $O$, and $T$ the tension at $P$ (tensions are tangential because the cable is *flexible*); $W$ the load supported by $s=\text{arc}\,OP$. Then considering the equilibrium of arc $OP$ under the forces $H$, horizontal, $W$ vertical, $T$ along the tangent at $P$, and the differential triangle $PRS$, we find $H:W:T=PR:RS:PS=dx:dy:ds;$ in particular $T^2=H^2+W^2$.]

31. Find the form of arc of a suspension bridge cable and the tensions.

[Practically, $W = cx$, and putting $H = ca$, from $H : W = dx : dy$ we find $a\,dy = x\,dx$ and $y^2 = x^2/2a$, a parabola. Let $(h, k)$ be one end of the cable; then $k^2 = h^2/2a$ or $a = h^2/2k$; $T = c\sqrt{(a^2 + x^2)}$, $= c\sqrt{(a^2 + 2ay)}$, $= c(a + y)$ approximately, if $y$ is small compared with $a$, *i. e.*, if $k$ is small compared with $h$.]

32. Find the tensions and form of arc of a cable with uniformly distributed load.

[Here $W = cs$, and putting $H = ca$, then $T = c\sqrt{(a^2 + s^2)}$. From $W : T = dy : ds$, we find $dy = s\,ds/\sqrt{(a^2 + s^2)}$, $y = \sqrt{(a^2 + s^2)} - a$, $T = c(a + y)$. If at one end $s = l$, $y = k$, $x = h$ then $k + a = \sqrt{(a^2 + l^2)}$, $a = (l^2 - k^2)/2k$, $= h^2/2k$ approximately if $k$ is small compared with $l$ so that $l$ is nearly straight. From $W : H = dy : dx$, we find

$$dx = a\,dy/s = a\,dy/\sqrt{(a + y)^2 - a^2};$$

$$x = a \log \frac{a + y - \sqrt{(a + y)^2 - a^2};}{a}$$

$$e^{\frac{x}{a}} = \frac{a + y - \sqrt{(a + y)^2 - a^2}}{a},$$

which gives

$$\frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) = a + y,$$

a catenary. Using the expansion of $e^{\frac{x}{a}}$ we have approximately $y = x^2/2a$, a parobola.]

### Curve Tracing

**87.** To trace the locus of $F(x, y) = o$, take a series of values of $x$, and for each value $x = a$ find from the equation the corresponding values $y=b,b'\ldots$, and plot the points $(a,b)$, $(a,b'),\ldots$ The points so plotted on each verticle line $x=a$ are the points where the several branches of the locus cross that line. When the vertical lines are close enough the form and continuation of each branch will be shown by its dotted construction. This is the primitive method; an improvement consists in drawing a short dash at each plotted point in the direction of the tangent, when fewer points are necessary. The tangent is drawn from its slope $dy/dx$ which is $-F_1(x,y)/F_2(x,y)$ where the numerator and denominator are the partial derivatives of $F(x,y)$ as to $x, y$, respectively; viz., since $F(x,y)$ remains zero as $x, y$ change continuously, therefore $dF(x,y)=o$, or by Art. 34 (a), $F_1(x,y)\,dx+F_2(x,y)\,dy=o$. A further improvement is to obtain an accurate idea of the general form of the locus from a systematic study of the equation, when it will be necessary to plot the locus with care only at a few critical points. The methods of such study will be considered in detail.

**88.** *Examine the equation for symmetry as to axes and origin.* The test of symmetry is that the substitution, for the co-ordinates $(x,y)$, of the co-ordinates of the symmetric point, in the equation of the locus, must leave the equation unaltered. *E.g.*, the locus of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is symmetric as to the $x$-axis because changing $(x,y)$, into $(x,-y)$, the symmetric point as to the $x$-axis, leaves the equation unaltered. Similarly, this locus is symmetric as to the $y$-axis because changing $(x, y)$ into $(-x, y)$ leaves the equation unaltered; and it is symmetric as to origin, because changing $(x, y)$ into $(-x, -y)$ leaves the equation unaltered. In general, the substitution, for the co-ordinates $x, y$, of the co-ordinates of the symmetrical point, in any equation, gives the equation of the symmetric locus; *e. g.*, the locii of $y^2=8x+12$ and $y^2=-8x+12$ found by replacing $(x, y)$ in the first by $(-x, y)$ are symmetric locii as to the $y$-axis since if any point $(a, b)$ satisfies the first, then $(-a, b)$ satisfies the seeond.

**89.** *Examine the equation for limits of real value of $x$ and $y$.* If $y$ is imaginary when $x$ lies between $a$ and $b$ then no part of the locus lies between the vertical lines $x=a$ and $x=b$; for although the imaginary value of $y$ is in such case an algebraic solution of the equation, and $(x, y))$ is a point of the locus in an algebraic sense, yet no point in the plane of representation corresponds to it. *E.g.*. in $x^2/a^2+y^2/b^2=1$, where $a, b$ are real numbers, if $x^2>a^2$ then $y$ is imaginary, and if $y^2>b^2$ then $x$ is imaginary; hence the locus lies between the vertical lines $x=\pm a$, and the horizontal lines $y=\pm b$.

**91.** *Directions to and at infinity.* The direction whose slope is the limiting value of $y/x=\tan\theta$ as the point $(x, y)$ approaches infinity on a distant branch is *the direction to infinity of that branch.* The direction *at* infinity of the branch is the direction given by the limit of the slope $dy/dx=\tan\phi$ at the distant point $(x, y)$ on the branch. This is identical with the direction to infinity; for $\lim y/x = \lim dy/dx$ by the theory of indeterminate forms, when both $x$ and $y$ approach infinity, and when $\lim y/x = o$ or $\infty$ in consequence of $y$ or $x$ approaching a finite limit then also $\lim dy/dx = o$ or $\infty$. *E.g.*, if for a finite value $x$ we have $y = \infty$, then $x'$ being near to $x$, $y'$ will be finite, and $\Delta y = y' - y = \infty$, so that $dy = \infty$ when $dx$ is infinite, and $dy/dx = \infty = y/x$. It appears that a distant branch with a limiting direction is very nearly a straight line of slope $\lim dy/dx = \lim y/x$. A spiral winding indefinitely around the origin and extending indefinitely outward is an example of a distinct branch with no direction to or at infinity.

**92.** Asymptotic Line. When all the points of a distinct branch approach more and more nearly coincidence with the distant points of a given straight line, such line is *the asymptote* of the branch. To have an asymptote, it is evident that the branch must have a direction to infinity; also, the tangent to the branch at a point approaching infinity must have this asymptote for its limiting position, so that the asymptote is a line *tangent to the branch at infinity*. If $(x, y)$ be a point on the branch approaching infinity, then excepting vertical branches for which $\lim y/x = \infty$, we shall have $\lim y/x = m$, the slope of the asymptote, and the ordinate of the asymptote that passes through the point $(x, y)$ on the branch will be $mx + b$, where $b$ is $y$-intercept of the asymptote. The condition that the branch approaches coincidence with the asymptotic line is that the difference of their ordinates to the same abscissa, or $(mx+b)-y$, opproaches zero when $x$ approaches infinity. Thus $b=\lim(y-mx)$. Denoting $y-mx$ by $q$, we have then to put $y=mx+q$ in the equation of the locus and in the resulting equation for $q$ in terms of $x$ find the limiting value of $q$ as $x$ approaches infinity, under the condition that $q$ is to be finite. If the equation between $q$ and $x$ is algebraic, we divide it by the highest power of $x$ and find the limiting equation by putting $x=\infty$. The value of $q$ from this equation is the required $y$-intercept of the asymptote.

**83.** For example find the directions to infinity and the asymptotes of $xy^2-x^3-2ay^2+a^3=o$. Divide this by $x^3$ and put $x=\infty$; if $y$ be supposed finite we obtain $o=o$, which finds no finite value of $y$ when $x=\infty$; if $y/x$ be supposed finite and equal to $m$, we find $m^2-1=o$, which gives two directions to infinity of slope 1 and $-1$ respectively. Divide the equation by $y^2$ and put $y=\infty$; if $x$ be supposed finite, we find $x-2a=o$. This is therefore a vertical direction to infinity whose asymptotic line is $x=2a$. To find the asymptotes of slope $m=1$ or $-1$, put $y=mx+q$ in the given equation, and it becomes, since $m^2=1$, $2(mq-a)x^2+(q^2-2amq)x-2aq^2+a^3=o$; dividing this by $x^2$, we see that if $q$ remains finite as $x$ increases indefinitely, we must have in the limit, $mq-a=o$ or $q=a/m=ma$. Thus $y=m(x+a)=\pm(x+a)$ is the equation of the asymptote of slope $m=\pm 1$.

(a) Show that, *the terms of highest degree in the equation of a locus, when equated to zero, give the equation of the lines* *to infinity through the origin;* viz., in the above example, $xy^2-x^3=o$, are such lines, etc.

**94.** *Examine the equation for regions of rising and falling branches (from left to right) and resulting crests and hollows.*

In other words, note from the equation where $y$ increases, where $y$ decreases, and where it is at a maximum or minimum value. If the equation does not show this readily, it is determined by the values of $(x, y)$ that make $dy/dx$ positive in the first case, negative in the second case, and where $dy/dx$ is changing sign in the other cases.

**95.** *Examine the equation for regions of concavity upward, concavity downward, and consequent points of inflection,*

In other words, note where the tangent turns counterclockwise as its point of contact advances to the right (which is shown by the slope $p=dy/dx$ increasing or by $dp/dx$ positive) where the tangent turns clockwise (which is shown by the slope $p=dy/dx$ decreasing or by $dp/dx$ negative) and where the tangent is changing direction of turning (which is shown by $dp/dx$ changing sign.)

**96.** *Examine the equation for multiple potnts, and trace the locus in the neighborhood of a multiple point.*

A multiple point is a point where two or more branches of the locus intersect; it is a double or triple point, *etc.*, according to the number of branches.

At such a point $p=dy/dx=-F_1(x,y)/F_2(x,y)$, (Art. 87) must be correspondingly multiple valued, which can only be (excluding discontinuity) when this fraction is $o/o$ for the point $(x, y)$. Thus a multiple point must be a solution of the simultaneous equations $F(x,y)=o$, $F_1(x,y)=o$, $F_2(x,y)=o$. For such a point we have by the theory of indeterminate forms, $p=-dF_1(x,y)/dF_2(x,y)$ which becomes, after replacing $dy/dx$ by $p$, and $x, y$ by their values at the multiple point, a quadratic for $p$. If this quadratic is determinate the point is a double point; and the branches intersect, or touch, or are imaginary, according as $p$ has two different or eqnal real values or two imaginary valaes. In the latter case the multiple point is an *isolated* point of the locus with no real point next to it. In general, if $(x_1, y_1)$ is the double point, we put $x=x_1+h$, $y=y_1+k$ in the equation of the locus, and trace the locus for small negative and positive values of $h$. This is, in effect, transforming the axes to parallel axes through the multiple point, with $h, k$ as co-ordinates to the new axes. It is easily shown that the terms of lowest degree in the resulting equation for $(h, k)$ when put equal to zero, will be the equation of the tangents at the multiple point, which is now made the origin; and this equation determines at once by its degree, the order of the multiple point. If $m$ be the slope of one of these tangents, we trace the branch to this tangent by putting $k=mh+q$ in the equation of the locus, aud determine from the resulting equation in $q$, $h$, the principle part of $q$ for small positive and negative values of $h$, and thence the position of the corresponding point $(h, k)$ above or below the tangent, according as $q$ is positive or negative.

**97.** Trace the locii of the following equations, taking convenient lengths for the constants.

$$2ay = x^2 - 2ax + ab;$$
$$3a^2y = x^3 - 3ax^2 + 3(a^3 \pm b^3)x + a^3c \qquad (b \gtreqq o)$$

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1; \quad \left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1; \quad \left(\frac{x}{a}\right)^2 \pm \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1;$$
$$a^2y^2 = a^2x^4 - x^6.$$

$y^2 = x^3/(2a - x)$, the *cissoid;* if $OP$ intersect the vertical line $x = 2a$ in $Q$, and the circle on the abscissa $2a$ as diameter in $R$, show that $OR = PQ$.

$$ay^2 = x^3; \; ay^2 = (x - a)^2(x - b) \qquad (a >, <, = b).$$

$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, the hypocycloid described by the rolling of a circle of radius $a/4$ inside a circle of radius $OA = a$, the tracing point being on the circumference of the rolling circle and on the axes when in contact with the fixed circle.

$y = 8a^3/(x^2 + 4a^2)$, the witch; draw a circle with vertical diameter $OA = 2a$; draw a line from $O$ to meet the circle in $R$ and the tangent at $A$ in $Q$, when the abscissa of $Q$ and ordinate of $R$ are the $(x, y)$ of a corresponding point $P$ of the witch.

$$x^3 + y^3 = 6ax^2; \quad x^3 + y^3 = 3axy;$$
$$x^3 + y^3 = a^3; \quad x^4 - 2a^2y = ay^3.$$

$(x^2 + y^2)^2 = a^2(x^2 - y^2)$, the lemniscate, if $r$, $r'$ be radii to $P$ from focii $F$, $F'$ on the $x$-axis such that

$F'O = OF' = a/\sqrt{2}$, then $rr' = a^2/2$; with polar radius $r$ from $O$, $r^2 = a^2 \cos 2\theta$.

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) = a \cosh \frac{x}{a}, \text{ the catenary.}$$

**98.** The same methods may be employed in tracing the locii of polar equations. In looking for symmetry the symmetric of $(r, \theta)$ has several forms that must be tried separately: *e.g.*, the symmetric points of $(r, \theta)$ as to $OX$ are $(r, -\theta)$, $(-r, \pi-\theta)$, *etc.* A direction to infinity is a value of $\theta$ that makes $r=\infty$, and the corresponding asymptote is found from the limit of the polar subtangent $OM' = r^2 d\theta/dr$, whose direction of measurement is $\theta - \pi/2$. A direction $\theta$ that makes $r=o$ is the direction of a tangent at the origin, since that is the limiting direction of the chord $OP$ as $P$ approaches $O$. The locus recedes or approaches the origin as $\theta$ increases if $r^2$ is increasing or decreasing, and it is concave towards or from the origin according as $\tan\psi = r d\theta/dr$ increases or decreases with $\theta$. The following equations are given to trace:

$r = a\theta$, the spiral of Archimedes; $r = ae^{\frac{\theta}{c}}$, the equiangular spiral.

$r = 2a \sin\theta$, $r = 2a \cos\theta$, $r = a \sec 2\theta$, $r = a/\theta$.

$r = a \sec \frac{\theta^2}{2}$; $r = a \sin 2\theta$, $r = a \cos 2\theta$; $r = a \sin \frac{\theta^3}{3}$.

$r^2 = a^2 \cos 2\theta$; $r = a(1 - \cos\theta)$, the cardioid.

$r = a + b \csc\theta$; $r = a(\sec 2\theta + \tan 2\theta)$;

$r = a^2 \csc\theta^2 + b^2 \sec\theta^2$.

$r = a \cos\theta + b \sin\theta$; $r = a \cos 2\theta + b \sin 2\theta$.

**Envelopes**

**99.** Let three variables $x, y, t$ be always connected by a given equation $F(x, y, t)=o$; then to a given point $(x, y)$ corresponds one or more numbers (the numbers of $P$) which are the solutions of the given equation for $t$ in terms of the given values of $(x, y)$. We assume every point of the plane to be so numbered by this equation. The equation $F(x, y, t)=o$ may be the equation of any locus we please in the plane by selecting for $t$ a proper functional value $t=f(x, y)$. In other words, consider any given locus in the plane, and select from point to point of that locus one of the numbers of each point so that this number varies continuously with the position of the point; then the continuous assemblage of such numbers form a definite function $t=f(x, y)$. It is obvious, on account of the multiplicity of thc numbers of each point, that it may be possible to find different functional values of $t$ such that for each, $F(x, y, t)=o$ shall be the equation of the same locus. Since $dF(x, y, t)=o$ on such curve, we have (using the notation of Art. 34 for partial derivations as to the first, second, and third variables)

(a) $F_1(x, y, t)\,dx+F_2(x, y, t)\,dy+F_3(x, y, t)\,dt=o.$

This is an equation for the slope $dy/dx$ at the point $(x, y)$ on the given curve, remembering that $dt$ is of the form $Ldx+Mdy$ depending upon the given locus.

**100.** *The n-curve.* The locus of all points having the same number $n$ is the $n$-curve. Its equation is $F(x, y, n)=o$. Thus the $o$-curve is $F(x, y, o)=o$, the

*1*-curve is $F(x, y, 1)=o$, *etc.* The slope of the $n$-curve is given by $t=n$, $dt=o$ in 99a and is

(a) $F_1(x, y, n)dx+F_2(x, y, n)dy=o$.

**101.** *The self-intersections.* The points $(x, y)$ that simultaneously satisfy $F(x, y, n')=o$ $F(x, y, n)=o$ may be called the $n'.n$ points, because they are each points of number $n'$ and $n$; they are the intersections of the $n'$-locus with the $n$-locus. The limiting positions of these intersections as $n'$ is taken nearer and nearer and to its limit $n$ are the $n.n$ points, or *self-intersections* of the $n$-locus. To find these $n.n$ points we must replace the $n'$-locus by another that always intersects the $n$-locus in the $n'.n$ points and only those, and that does not reduce to the $n$-locus itself when we put $n'=n$. This locus is given by

$$[F(x, y, n')-F(x, y, n)]/(n'-n)=o.$$

since any point $(x. y)$ that satisfies this equation and $F(x, y, n)=o$, will also satisfy $F(x, y, n')=o$ and so be an $n'.n$ point, and conversely every $n'.n$ point satisfies the above equation. Taking $n$ as the original value and $n'$ as the new value of $t$, so that the equation is, $\Delta_t F(x, y, t)/\Delta t=o$, we see that its limit is $F_3(x, y, n)=o$. Hence

(a) *The self-intersections of an n-locus are its intersections with the locus of the partial derivative of its equation as to its number (regarded as the original value of the variable t)*; *i. e.*, the $n.n$ points are the values of $(x, y)$ that simultaneously satisfy $F(x, y, n)=o$, $F_3(x, y, n)=o$.

**102.** By solving the preceding simultaneous equations for $(x, y)$ we find the co-ordinates of self-intersection of the $n$-locus each in terms of the number $n$ of that locus; then by giving $n$ all values, we find the assemblage or locus of all $n.n$ points for all values of $n$, or *the locus of self intexsections.* By eliminating $n$ between the above simultaneous equations (by solving one for $n$ and substituting its value in the other) we evidcntly obtain the equation of the locus of self intersections in terms of $(x, y)$ alone. We may select the variable function $t=f(x, y)$, so that $F(x, y, t)=o$ is the equation of the locus of self-intersections, viz., $f(x,y)$ is a solution of $F_3(x, y, t)=o$ for $t$ in terms of $x, y$. The slope of the locus of self-intersections is then given by 99(a) which reduces since $F_3(x, y, t)=o$ to

(a) $F_1(x, y, t)\,dx+F_2(x, y, t)\,dy=o.$

**103.** *The multiple points of an n-locus are points of self-intersection of that locus.* For at a multiple point $(x, y)$ of the $n$-locus, $F(x, y, n)=o$, we have also $F_1(x, y, n)=o$, $F_2(x, y, n)=o$ (to make $dy/dx=o/o$). Thus, substituting $t=n$ in 99(a) which is true for all values of $(x, y)$ of number $t$, even when $dt$ is not zero, we find that $F_3(x, y, n)=o$; *i.e.*, by 101a, the multiple point $(x, y)$ on the $n$-locus is a point of self-intersection. We divide the locus of self-intersections into that of ordinary points and that of multiple points of the $n$-locus.

**104.** *The locus of ordinary self-intersections is met tangentially at each point by the n-locus on which that point is a self-intersecton.* For the slope of the locus of self-intersections at such point $(x, y)$ whose number is $n$, is found by making $t=n$ in 102a, and since the differential co-efficients are definite non-zero values (because the point is an ordinary point on the $n$-locus) therefore this slope is the same as the slope of the $n$-locus, given by 100a; *i.e.*, the two locii meet tangentially. This result does not hold on the multiple point locus, since then $F_1(x, y, n)=o$ $F_2(x, y, n)=o$; and in general $dy/dx=o/o$ signifies that the value of $dy/dx$ at such limiting point depends upon the manner of approach of $(x, y)$ to their limiting values, and is otherwise absolutely indeterminate. Now on the $n$-curve, we are to find the limit of $-F_1(x, y, n)/F_2(x, y, n)$ as $(x, y)$ approaches its limit on the locus $F(x, y, n)=o$; and on the multiple point locus we are to find the limit of $-F_1(x, y, t)/F_2(x, y, t)$ as $x, y, t$ approach their limiting values wherein $t$ is a variable approahing $n$ and conditioned by $F_3(x, y, t)=o$. These are certainly different methods ot approach and give in general different limiting values for $dy/dx$. *E.g.*, on the $n$-curve $(y-n)^2=(x-a)^3$ the point $x=a$, $y=n$ is a multiple point whose locus, as $n$ varies through all values, is the vertical line $x=a$. This is the only locus of self-intersections, as may be shown by eliminating $n$ between this equation and the $n$-derivative, $y-n=o$; and its slope is $dy/dx=\infty$ at every point. On the contrary, the $n$-curve has two branches $y-n=\pm(x-a)^{\frac{3}{2}}$ that meet to form the multiple point at $x=a$, $y=n$, and on these branches $dy/dx=\pm\frac{3}{2}(x-a)^{\frac{1}{2}}$ whose limit when $x$ approaches $a$ is zero, *i.e.*, the slope of every $n$-curve is zero at its multiple point, and it therefore meets the multiple point locus everywhere at right angles—quite the reverse of tangential meeting. In general any definite motion of a curve with a multiple point, as a lemniscate which is a figure 8, generates a system of $n$-curves, in which $n$ may be taken as the time at which the generating curve is an $n$-curve; and such motion can be so determined that the locus of the multiple point shall meet the $n$-locus at any angles we please, constant or varying with the locus.

**105.** *If a given locus is met tangentially at every point by an n-locus through that point, then it is a locus of self-intersections.* For, take the equation of the given locus as $F(x, y, t)=o$ where the variation of $t$ with $(x, y)$ is determined by the condition that $t=n$ at the point of tangency $(x, y)$ of an $n$-locus. The condition of tangential meeting at $(x, y)$ is then that the slope given by 99a when $t=n$ is identical with the slope given by 100a. Thus, making $t=n$ and subtracting, remembering that $t$ is a variable so that $dt=o$ for any continuous series of values of $(x, y)$ on the given locus is inadmissible, we find $F_3(x, y, n)=o$ *i.e.*, any point $(x, y)$ of the given locus is a self-intersection. The complete locus that satisfies the above condition of tangency or envelopment by the $n$-curves will be called the *envelope* of the system of $n$-curves. This envelope will not in general include the multiple point locus and will be simply the locus of ordinary self-intersections.

**106.** Let $F(x, y, t)=o$ be the equation connecting the volume $x$, pressure $y$, and temperature $t$ of a unit mass of gas; then the $n$-curves of this equation are the so-called *isothermal lines* of the gas of temperature $n$.

For the so-called perfect gas $xy=at$, and the isothermals are hyperbolas. An intersection of two isothermals of different temperatures implies an unstable condition of the gas, and is in general impossible.

**107.** Let $F(x,y,z)=o$ be the equation of a surface in which $XOY$ is a horizontal plane, and $z$ is the height at the point $x, y$; then the *n-curve* $F(x,y,n)=o$ is *the contour line*, on the plane $XOY$, of points on the surface of altitude $n$. In contour maps, we have also no intersection of contour lines of different altitudes, because to each point $(x, y)$ corresponds only one altitude number.

**108.** Find the envelope of the following systems of curves, $m$, $p$, $q$, $t$, etc., denoting variable parameters. For straight line systems draw also a sufficient number of lines of each system to show the envelope graphically.

(a) $y=mx+a/m$; $\qquad y^2=4ax$

(b) $(y-mx)^2=a^2m^2\pm b^2$; $\qquad x^2/a^2\pm y^2/b^2=1$

(c) $x\cos t+y\sin t-a=o$; $\qquad x^2+y^2=a^2$

(d) If points $Q$, $R$, move uniformly along straight lines $OA$, $OB$, show that $QR$ envelopes a parabola.

$[x/t+y/(at+b)=1$; $(ax+y)^2+2b(ax-y)+b^2=o]$

(e) Find the envelope of a line of constant length ($a$) moving with its ends in the axes. $[x/p+y/q=1$, $p^2+q^2=a^2$, treat $p$, $q$ as functions of $t$, then on the envelope $x\,dp/p^2+y\,dq/q^2=o$, $pdp+qdq=o$, which gives $x/p^3=y/q^3$ by eliminating $dp$, $dq$. To eliminate $p, q$ denote for the moment the common value of the members by $r$; then substituting in preceding equation gives $r=1/a^2$, etc. Ans. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}]$

(f) Particles are started from the origin with equal speeds in varying directions $\theta$, in a vertical plane; find the envelope of their paths.

[$x=at\cos\theta$, $y=at\sin\theta-gt^2/2$, and the path is $y=x\tan\theta-gx^2/2a^2\cos\theta^2$. Ans. $y=a^2/2g-gx^2/2a$]

(g) Find the envelope of the variable ellipse $x^2/p^2+y^2/q^2=1$ of constant area $\pi a^2$ $(pq=a^2)$; also the one of fixed director circle $(p^2+q^2=a^2)$; also the one in which $p+q=a$.

Ans. $4x^2y^2=a^2$; $(x\pm y)^2=a^2$; $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

(h) Find the envelope of the normal to the parabola $y^2=4ax$. [$y=m(x-2a)-am^3$; $27ay^2=4(x-2a)^3$]

(i) Show that the self-intersection of the normal of a given curve at $x, y$ is the center of curvature of the curve at $(x, y)$.

[$\mathbf{y}-y=-p(\mathbf{x}-x)/p$ where $x$ is the variable parameter, $y$ a function of $x$ given by the equation of the curve, and $p=dy/dx$. Ans. $\mathbf{x}=x-p(1+p^2)/q$, $\mathbf{y}=y+(1+p^2)/q$, where $q=dp/dx$, and these are the co-ordinates of the center of curvature (Art. 84.)]

(j) Show that the normal of a curve regarded as a rigid line terminating in the corresponding point $(x, y)$ of the curve rolls on its envelope without slipping. [$\mathbf{x}=x-R\sin\phi$ gives $d\mathbf{x}=dx-R\cos\phi\, d\phi-\sin\phi\, dR$ $=-\sin\phi\, dR$, since $R\cos\phi\, d\phi=ds\cos\phi=dx$. But if $\mathbf{s}$ be the arc of the envelope. $d\mathbf{x}=d\mathbf{s}\cos(\phi+\frac{\pi}{2})=-d\mathbf{s}\sin\phi$. Hence $d\mathbf{s}=dR$, and hence $\Delta\mathbf{s}=\Delta R$, or increase of distance $R$ on the normal between $x, y$ and point of contact $\mathbf{x}$, $\mathbf{y}$ equals arc rolled over.]

# CHAPTER IV

## Integration

**109.** *Variation.* To give a variable $x$, a series of values one after another, determines *a variation of* $x$. By any variation of $x$, a variation of $\Delta x$ is also determined, and any function of $x$ and $\Delta x$ also varies through the same number of values as $\Delta x$. *E.g.*, $x=1, 3, 7, 10$, is a variation of $x$ from 1 to 10, in which $\Delta x=2, 4, 3$ where successively $\Delta x=2$ corresponds to $x=1$, $\Delta x=4$ to $x=3$, $\Delta x=3$ to $x=7$; we have also $x\Delta x=2, 12, 21$; $x^2\Delta x=2, 36, 147$; $(x^2+x)\Delta x=4, 48, 168$; $(x^2+x\sin\Delta x)\Delta x=2+\sin 2,\ 36+12\sin 4,\ 147+21\sin 3$; and so on. Another variation from 1 to 10 is $x=1, 2, 4, 5, 7, 9, 10$; in which $\Delta x=1, 2, 1, 2, 2, 1$; $x\Delta x=1, 4, 4, 10, 14, 9$; $x^2\Delta x=1, 8, 16, 50, 98, 81$; $(x^2+x)\Delta x=2, 12, 20, 60, 11, 90$; etc. Another variation $x=1, 2, 3, 4, 5, 6, 7, 8, 9, 10$, in which $\Delta x=1$, always, is on that account called uniform variation. The general variation of $n$ changes from $a$ to $u$ will be denoted by $x=a, b, c, \ldots\ l, u$, in which $\Delta x=b-a,\ c-b,\ \ldots\ u-l$.

The final value $x=u$, may be considered either as a constant or a variable,—in the latter case $u$ is another symbol for an original value of $x$, regarded as reached by variation from the initial value $a$; and it may be considered as used merely to prevent confusion between the final value and the intermediate values.

**110.** *Summation.* The symbol $\Sigma_a^u f(x, \Delta x)=$ "the sum from $a$ to $u$ of $f(x, \Delta x)$" stands for the sum of the values of $f(x, \Delta x)$ in a variation of $x$ from $a$ to $u$. *E.g.*, when $x=1, 3, 7, 10$, $\Sigma_1^{10} x\Delta x=35$, $\Sigma_1^{10} x^2 \Delta x=185$, $\Sigma_1^{10}(x^2+x)\Delta x=220$. When $x=1, 2, 4, 5, 7, 9, 10$, the same sums are 42, 254, 296, respectively. When $x=1, 2, 3, 4, 5, 6, 7, 8, 9, 10$, the same sums are 45, 285, 330, respectively. A sum will therefore in general depend upon the variation of $x$ between its assigned limits. If $u$ is a variable in the above sum, such sum is what is called an *imperfect* function of $u$, that is, it depends upon, but is not determined by the value of $u$; it requires the method of variation of the variable $x$ from its initial value $a$ to its final value $u$ to be also assigned. This imperfect function has, however, a definite difference $f(u, \Delta u)$, since changing $u$ to $u+\Delta u$ changes the sum by the term $f(u, \Delta u)$; *i.e.*, $f(x, \Delta x)$ is the difference of the sum for any final value of $x$. It is in general an *imperfect* difference *i.e.*, not the difference of a function.

**111.** Theorem 1. *The sum from a to u of the difference of a function is equal to the change from a to u of the function.* In symbols, $\Sigma_a^u \Delta\phi x = \Big|_a^u \phi x = \phi u - \phi a$.

For the successive values of $\Delta\phi x$ in any variation $x=a, b, c, \ldots l, u$, are $\phi b-\phi a$, $\phi c-\phi b$, $\ldots$ $\phi u-\phi l$, and adding them, the intermediate values $\phi b, -\phi b$, $\phi c, -\phi c, \ldots \phi l, -\phi l$, all cancel, leaving only $\phi u-\phi a$ for the sum. This result may also be stated as follows: *The sum of the successive changes of value of a function is* *its total change of value.* As an illustration verify that $\Sigma_1^{10}\Delta . x^2 = \Sigma_1^{10}(2x\Delta x + \Delta x^2) = \Big|_1^{10} x^2 = 99$, for each of the variations of Art. 109.

(a) It appears that $\Sigma_a^u \Delta\phi x$ is a definite function of $u$, *i. e.*, independent of the variation from $a$ to $u$, and that it is "a function whose difference is $\Delta\phi u$"; *i. e.*, $\Sigma = \Delta^{-1}$.

**112.** As exercises, show that for any given variation $x = a, b, c, \ldots g, h, i, \ldots l, u$, we have:

(a) $$\Sigma_a^u [c' f(x, \Delta x) + c'' f_1(x, \Delta x)]$$
$$= c' \Sigma_a^u f(x, \Delta x) + c'' \Sigma_a^u f_1(x, \Delta x).$$

In words, *the characteristics of summation for a given variation, is distributive over a sum and commutative with a constant factor.*

(b) $\Sigma_a^u f(x, \Delta x) = \Sigma_a^h f(x, \Delta x) + \Sigma_h^u f(x, \Delta x)$.

(c) $\Sigma_a^u f(x, \Delta x) = \Sigma_u^a f(x, -\Delta x)$

**113.** *Continuous variation.* Let there be an unending number of successive variations from $a$ to $u$, of greater and greater number of changes, and formed according to some definite law. In the variation of $n$ changes, every value of $\Delta x$ will be smaller* than some number $h_n$; $h_n$ cannot be smaller than $(u-a)/n$, the value of

*We use the terms "larger" and "smaller" with reference to comparative magnitudes, which are positive numerical values. The magnitude of any number is the positive square root of its square, if it is a rea number, and in general, it is the positive square root of the sum of the squares of the real components of the number.

$\Delta x$ when the variation is uniform, but it is otherwise any number we please according to the variation. If we form an unending series of variations in such a way that the superior limit $h_n$, of every $\Delta x$ in the variation of $n$ changes, approaches zero as a limit as $n$ approaches infinity, then such a series will be called *an approach to a corresponding continuous variation from a to u.* In other words, in the variation of $n$ changes, when $n$ is very large, the values of $\Delta x$ will each be very small, so that the variation is approximately continuous. There are many kinds of continuous variation from $a$ to $u$ according to the defining series of variations of greater and greater number of changes and smaller and smaller values for each change. Uniform continuous variation is the limit of a series of uniform variations, in which $h_n=(u-a)/n$. In a non-uniform variation, however nearly continuous, some changes may be very many times larger than other changes, although every change may be very small.

**114.** *Continuous summation—Integration.* The limit of the sum of a difference $f(x, \Delta x)$, as the variation of $x$ from its initial to its final value approaches continuous variation, is a *continuous sum*, or *integral.* For example, we will find the values of the integrals, $\lim \Sigma_a^u x\Delta x$, $\lim \Sigma_a^u x^2 \Delta x$, $\lim \Sigma_a^u x\Delta x^2$ for uniform continuous variation. The uniform variation of $n$ changes from $a$ to $u$ is $x=a,\ a+h,\ a+2h, \dots\ a+(n-1)h, u,$ where $nh=u-a=n\Delta x$. With this variation the above sums are, since $1+2+\dots+n-1=n(n-1)/2$, and $1^2+2^2+\dots+(n-1)^2=n(n-1)(2n-1)/6$,

the first $=h[na+n(n-1)h/2]$
$=(u^2-a^2)/2-h(u-a)/2;$

the second $=h[na^2+n(n-1)ah+n(n-1)(2n-1)h^2/6]$
$=(u^3-a^3)/3-h(u^2-a^2)/2+h^2(u-a)/6;$

the third $=h$ times the first.

Hence the limits of these sums for $n=\infty$ or $h=o$ are $(u^2-a^2)/2$, $(u^3-a^3)/3$, and $o$, respectively.

**115.** We shall show that under certain conditions an integral or continuous sum exists in which different methods of approach to continuous variation have no effect upon its value; and we shall assign a notation that embraces in compact form the fundamental facts and ideas of such limiting sum, and determine, by a fundamental theorem, shorter methods for evaluating integrals than the full process, which is complicated even for uniform variation, (See Art. 114).

We consider in the first place, only sums $\Sigma_a^u f(x, \Delta x)$ in which the proportional difference $Nf(x, \Delta x)$ approaches a definite differential in terms of $x, dx$ when as usual, $N$ approaches infinity and $\Delta x$ approaches zero so that $N\Delta x$ approaches any assigned value $dx$.

There are, in fact, no general methods for determining whether an integral exists or not when no such differential exists. As in Arts. 20, 21 the differential, $\lim Nf(x, \Delta x)$ is of the form $\phi' x . dx$ when $x$ is a real variable, and may be so whether $x$ is real or imaginary. *E.g.*, $\lim Nx^2 \Delta x = x^2 dx$, $\lim N(x^2 + x \sin \Delta x) \Delta x = x^2 dx$, $\lim N\Delta (x^3/3) = x^2 dx$. As in these three examples, so in general, many different differences will lead to the same differential, and some of those differences must be perfect when the differential is perfect.

**116.** *Lemma. An integral over any range of variation is zero when the differential pertaining to the integral is identically zero.* In symbols, $\lim \Sigma_a^u f(x, \Delta x) = o$, *when* $\lim Nf(x, \Delta x) = o$, *identically.* For let $x$ be that value in any variation of $n$ changes which corresponds to *the largest term* $f(x, \Delta x)$ among the $n$ terms of this type; then the sum of the $n$ terms is certainly not larger than $nf(x, \Delta x)$, (the magnitude of a sum cannot exceed the sum of the magnitudes of its terms); but by hypothesis, and for the special case $N = n$, every product of the type $nf(x, \Delta x)$ approaches zero when $n$ approaches infinity and $\Delta x$, zero; and hence the given sum, that is always not larger than one of these products, must approach zero.

**117.** Theorem 2. *Integrals with identical differentials are also identical, for any same method of approach to continuous variation.* In symbols, *if* $\lim Nf(x, \Delta x) = \lim N\phi(x, \Delta x)$, then $\lim \Sigma_a^u f(x, \Delta x) = \lim \Sigma_a^u \phi(x, \Delta x)$.

For the difference of these integrals is the limit of the difference of the corresponding sums, which is $\lim \Sigma_a^u [f(x, \Delta x) - \phi(x, \Delta x)]$, by Art. 112a; and this integral is zero by the preceding lemma, since by hypothesis $\lim N[f(x, \Delta x) - \phi(x, \Delta x)] = o$ identically. While the two integrals are identical for any same *path of integration* from *a to u*, *i.e.*, for the same method of approach to continuous variation from $a$ to $u$, yet each may change value with change of path of integration.

**118.** *Notation.* Since the value of the integral depends only upon the corresponding differential and the path of integration, whether the difference of the sum has one or another of the many values whose proportionals approach the given differential, therefore the integral is named most definitely as "*the integral of the differential over the given path.*" Since we can write $\lim \Sigma_a^u f(x, \Delta x) = \lim \Sigma_a^u N^{-1}.\lim Nf(x, \Delta x)$ by multiplying by $N^{-1}N = 1$ at each stage of the approach, we therefore find $\lim \Sigma_a^u N^{-1} = \int_a^u$ say, as the characteristic of integration of the differential from $a$ to $u$; and if $\lim Nf(x, \Delta x) = \lim N\phi(x, \Delta x) = \ldots = \phi' x dx$, the identical integrals $\lim \Sigma_a^u f(x, \Delta x) = \lim \Sigma_a^u \phi(x, \Delta x) = \ldots$, are each expressed by $\int_a^u \phi' x dx =$ "integral from $a$ to $u$ of $\phi' x dx$."

**119.** Theorem 3. *The integral of the differential of a function is independent of the path of variation between given limits, and is equal to the total change of value of the function between the limits.*

In symbols, $\int_a^u d\phi x = \Big|_a^u \phi x.$

For since $\lim N\Delta\phi x = d\phi x$, therefore by the notation established by Th. 2, and by Th. 1,

$$\int_a^u d\phi x = \lim \Sigma_a^u \Delta\phi x = \lim(\phi u - \phi a) = \phi u - \phi a.$$

From this result follows

(a) $\int_a^u d\phi x =$ "*a function whose differential is*"

*$d\phi u = d^{-1}. d\phi u$, and that vanishes when $u=a$; i.e., $\int = d^{-1}$.*

Since $d = \lim N\Delta$ therefore, formally, $d^{-1} = \lim \Delta^{-1} N^{-1} = \lim \Sigma N^{-1} = \int$, from $\Sigma = \Delta^{-1}$. This result shows that the formal relations of $d, \int$, are consistent, *i. e.*, in accord with established facts.

**120.** *Inverse Prin.* 1. If $d\phi x = d\psi x$ when $x$ varies, then is $\phi x - \psi x$ a constant for variations of $x$.

For from the given identity and Th. 2

$$\int_a^u d\phi x = \int_a^u d\psi x$$

*i. e.*, by Th. 3, $\phi u - \phi a = \psi u - \psi a$, or $\phi u - \psi u = \phi a - \psi a$, a constant for variations of $x = u$.

**121.** The preceding results extend to variations, sums and integrals, in any number of variables. Thus a variation of $(x, y)$ will be a series of sumultaneous changes of $(x, y)$ from given initial values $(a, b)$ to any final values $(u, v)$; such variation determins a series of values of $(\Delta x, \Delta y)$ and a series of values of any function $f(x, \Delta x, y, \Delta y)$. There is no change in any of the preceding theorems and proofs except the slight changes consequent upon the introduction of the additional letters required for the values, changes of values and limits of the additional variables. The differentials in real variables $x, y$, etc., are of the forms

$$f'x \,.\, dx, f_1(x, y)\, dx + f_2(x, y)\, dy, \text{ etc.}$$

The differential $f'x\,dx$ where $f'x$ is a continuous function of $x$ is always a perfect differential $dfx$. *E.g.*, if $f'x$ is real, then draw the curve $y=f'x$, when the ordinate area, $fx$, of that curve from $x=a$ to any final value ot $x$, is a function of $x$ whose differential has been shown to be $dfx=y\,dx=f'\,x\,dx$, (Art. 77). The differential in more than one real variable is not, however, in general perfect, since this requires that the differential co-efficients be partial derivatives of a given function of the variables. The differentials in two real variables $x, y$ include differentials in an imaginary variable $z=x+y\sqrt{-1}$. When the differentials are imperfect then their integrals between given limits $(a, b)$, $(u, v)$ depend upon the manner of continuous variation, which is called *the path of integration.* When the differentials are perfect their integrals are independent of the paths of integration and functions of the final values of the variables (the initial values being given constants). A path of integration is determined when the corresponding values of the variables are determined in terms of one real variable, since this reduces the differential to a perfect differential in that one variable.

**122.** As illustrations, take $(x, y)$ as the co-ordinates of a point $P$ in the plane $XOY$, then the path of integration is shown by a path of $P$ from its initial to its final position. In this case the imperfect differential $y\,dx$ is the differential area described by the ordinate $y$, and $\int y\,dx$ along a given path is the total, continuously described, ordinate area of the path. For, $P, P'$ being two successive positions $(x, y)$ $(x', y')$ of the point on the path between its initial and final positions, then the ordinate area $PLL'P' = y_1 \Delta x$, where $y_1$ is some ordinate between $y = LP$ and $y' = L'P'$, is the typical difference $f(x, \Delta x, y, \Delta y)$ whose sum is the total ordinate area. Thus $\Sigma_{a,b}^{u,v} y_1 \Delta x$ denotes the total ordinate area, described by any $n$ successive changes along the given path. As we take $n$ larger and larger, such area is described more and more nearly continuously, *i.e.*, by sums of smaller and smaller differences, so that $\lim \Sigma_{a,b}^{u,v} y_1 \Delta x = \int_{a,b}^{u,v} y\,dx$ represents the result of continuous summation of ordinate areas described by continuous motion along the path. If along the given path $y = \phi' x$, then the integral area becomes $\int_a^u \phi' x\,dx$ which can be determined when a function $\phi x$ can be found such that $d\phi x = \phi' x\,dx$ viz., it will be $\Big|_a^u \phi x = \phi u - \phi a$. The path must be given in order to evaluate $\int y\,dx$. Another imperfect differential is the differential of radial area, $\frac{1}{2}(x\,dy - y\,dx)$; the integral of this between given limits is the total area described by the radius $OP$ from the initial to the final position of $P$, and it requires the path to be known, such as by "$y$ a given function of $x$" or "$x$, $y$ given functions of $\theta$," etc., before the integral is determinate. On the contrary, the sum of the ordinate and radial area described by $OPL$ is independent of the path, and is $\frac{1}{2}(uv - ab)$, since it is the integral, $\frac{1}{2}\int(x\,dy+y\,dx)=\frac{1}{2}\int d(xy)$, the integral of a perfect differential.

**123.** As a physical illustration, the amount of heat required to expand a gas from volume $x$ and pressure $y$ to volume $x'$ and pressure $y'$ depends upon the path or series of continuous changes from the condition $(x, y)$ to the condition $(x', y')$. The differential amount of heat is the amount that would be required to change from the condition $(x, y)$ to the condition $(x+dx,\ y+dy)$ if changes continued as at $(x, y)$, and is a quantity $L\,dx+M\,dy$, where $L, M$ are functions of $(x, y)$; when $dx, dy$ are very small this is *the principal part* of any of the actual amounts of heat required to make the change. Because the amount of heat absorbed varies with the path, this differential cannot be a perfect differential.

**124.** *Potential.* On the contrary the work done by a given natural field of force in displacing a given particle along any path is independent of the path when the terminal and initial points are the same. If $X, Y, Z$ be the components of the force in the directions of the axes of $x, y, z$ acting on the particle in the position $P=(x, y, z)$, then, the differential work for any displacement $ds=$ sum of the differential work of each component $=X\,dx+Y\,dy+Z\,dz$, which must be *a perfect differential.*

It thus appears that $X, Y, Z$ must be the partial derivatives of some function $\phi(x, y, z)$ as to $x, y, z$, and $-\phi(x, y, z)$ is called *the potential of the field* on the particle $P$. The existence of such a function was discovered by Lagrange; several years later Greene pointed out that it represented potential energy, or energy of position in the field with reference to the initial position; *viz.*, $-\phi(x, y, z)=c$ is the equation of the surface of potential $c$, and $c$ is the amount of work that will be done by the field in moving the particle along any path from this surface to the zero potential surface, *viz.*,

$$\int dw = \int d\phi(x, y, z) = \Big| \phi(x, y, z) = c.$$

EXAMPLES.

1. Find the sums from $x=o$ to $x=4$ of $x^2\Delta x$, $(x^2+\Delta x)\Delta x$, $\Delta . x^3/3$, for uniform variations of 2, 4, and $n$ changes. Also represent in each case the terms of the sum by rectangles in the plane $XOY$, of ordinates $y=x^2$, $y=x^2+\Delta x$, $y=\Delta(\frac{1}{3}x^3)/\Delta x$ respectively, and base $\Delta x$. (Representation by ordinate areas).

2. Show that the limit of each sum in Ex. 1 for $n=\infty$ is $64/3$; also verify that the proportionals of the differenences approach the same differential $x^2dx$. Show that in the ordinate area representation, the common limit of these sums is the ordinate area of the parabola $y=x^2$ from $x=o$ to $x=4$.

3. Find $\lim \Sigma_1^4 x\Delta x$ by uniform variation; also geometrically by its ordinate area representation; also by the fundamental theorem of integration (Art. 119).

4. Find by direct integration the functions whose differentials are $u\,du$, $u^2\,du$, and that vanish when $u$ vanishes.

Ans. $u^2/2$, $u^3/3$. (See Art. 114).

5. Verify by ordinate areas, that $\int_a^u x\,dx = \frac{u^2 - a^2}{2}$; also by Th. 3 and $d\,.\,x^2/2 = x\,dx$.

6. Verify by ordinate area that $\int_0^a \sqrt{(a^2 - x^2)}\,dx = \pi a^2/4$; also by Th. 3 and Ex. 14 p. 43.

7. Verify by ordinate area that $\int_0^u \sqrt{(a^2 - x^2)}\,dx = \frac{1}{2}u\sqrt{(a^2 - u^2)} + \frac{a^2}{2}\sin^{-1}\frac{u}{a}, = d^{-1}\,.\,\sqrt{(a^2 - u^2)}\,du.$

8. If the ordinates of two curves to any same abscissa are in a constant ratio $b/a$, then their ordinate areas between the same bounding ordinates are in the ratio $b/a$.

9. If parallel chords between two curves vary as their distances from a fixed point, then the area of a segment between two chords as bases is that of the rectangle on the altitude and the half sum of the bases.

[A chord at distance $x$ is $cx$, and the area between chords at distances $x = a$, $x = u$ is $\int_a^u cx\,dx = (u - a)(cu + ca)/2$.]

10. If the areas of parallel sections of a tubular surface vary as the squares of their distances from a fixed point, find the volume between two parallel sections in terms of the bases $b_1$, $b_3$, middle section $b_2$, and altitude $h$.

Ans. $h(b_1 + 4b_2 + b_3)/6$.

11. Find the volume of a cone or pyramid in terms of its base and altitude, and also the distance of its center of volume from the base.

Ans. $bh/3$, $h/4$.

12. Find the volume of a hemisphere of radius $a$, and the distance of its center of volume from its base.

$$V=\int_o^a \pi(a^2-x^2)\,dx=\frac{2\pi a^3}{3};$$

$$\text{mom } V/V=\frac{3}{2a^3}\int_o^a x(a^2-x^2)\,dx=\frac{3a}{8}$$

13. Find the moment of inertia of a right circular cylinder of altitude $c$ and radius $a$, about its axis.

$$\int_o^a x^2\,.\,2\pi cx\,dx=\frac{\pi c a^4}{2}$$

14. A wedge is made from a right circular cylinder of radius $a$ and altitude $h$, by plane sections through a diameter of one base and the tangents to the other base that are parallel to such diameter; find the volume of the wedge. [A plane perpendicular to the diameter at distance $x$ from the axis cuts each side piece that is taken off to make the wedge in a triangle whose area is to $ah/2$ as $a^2-x^2$ is to $a^2$, by similar triangles; thus $V=\pi a^2 h-4\int_o^a \frac{h}{2a}(a^2-x^2)\,dx=a^2h(\pi-\frac{4}{3})$.]

15. Find the volume common to two circular cylinders of common upper base and tangent lower bases.

$$2\int_o^a \frac{h}{a}(a^2-x^2)\,dx=\tfrac{4}{3}a^2h.$$

16. The axes of two right circular cylinders intersect at right angles; find the included volume, and the surface.

17. A sphere of radius $a$ is charged with $4\pi a^2\lambda$ units of electricity ($\lambda$ per unit area); find its potential at a point $C$ whose distance is $c$ from the center.

Draw the diameter $OC$, and let $CP=r$, $<OCP=\theta$ where $P$ is any point of a small circle of the sphere abont $OC$ as axis. The charge of the zone between the circle $P=(r,\theta)$ and the circle $P'=r', \theta'$ (whose altitude is $\Delta . r\cos\theta$) is $2\pi a\lambda\Delta . r\cos\theta$ and its potential at $C$ is $2\pi a\lambda\Delta(r\cos\theta)/r_1$ where $r_1$ is an average disiance between $r$ and $r'$. Since $r^2+c^2-2rc\cos\theta=a^2$, therefore $r\,dr=c\,d . r\cos\theta$, and the differential potential is $2\pi a\lambda d(r\cos\theta)/r=2\pi a\lambda\, dr/c$. When $c>a$ then $r$ changes from $c-a$ to $c+a$ giving the total potential $4\pi a^2\lambda/c$. When $c<a$ then $r$ changes from $a-c$ to $a+c$ giving the total potential $4\pi a\lambda$.

18. Simpson's Rule. When given the end chords $y_1$, $y_3$, the middle chord $y_2$, and the distance $h$ between chords, of an area between parallel chords, the generally best approximate value of the area from these data is $h(y_1+4y_2+y_3)/3$.

Any chord $y$ is a function of $x$, its distance from the middle chord, say $y=a+bx+cx^2+\ldots$, a convergent series. The three given chords can determine at most only three co-efficients, *i. e.*, the best approximation is to determine $y$ to three terms, and neglect the others as probably very small. From $y=y_1, y_2, y_3$, when $x=-h, o, h$, we have $y_1=a-bh+ch^2$, $y_2=a$, $y_3=a+bh+ch^2$, and consequently $y_1+y_3=2a+2ch^2$. The required area is

$$\int_{-h}^{h} y\,dx=\frac{h}{3}(6a+2ch^2)=\frac{h}{3}(y_1+4y_2+y_3)$$

When given 5 equi-distant chords $y_1$, $y_2$, $y_3$, $y_4$, $y_5$, then from the first and last three, the area is by this rule $h(y_1+4y_2+2y_3+4y_4+y_5)/3$, and so on. This rule applies when the $y$'s are the areas of equidistant parallel sections of a volume, *etc.*

10. If $y_1, y_2, y_3, y_4$, are four parallel chords of an area at equal distances $2h$, then $3h(y_1+3y_2+3y_3+y_4)/8$ is generally the best approximation to the area from chord $y_1$ to chord $y_4$.

[If $y$ be the chord at distance $x$ from the central chord of the area, then the best assumption is $y=a+bx+cx^2+ex^3$, etc.]

20. $$\int_0^{\frac{\pi}{2}} \sin x^{2n}\, dx = \int_0^{\frac{\pi}{2}} \cos x^{2n}\, dx = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots\ \ 2n}\cdot\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin x^{2n+1}\, dx = \int_0^{\frac{\pi}{2}} \cos x^{2n+1}\, dx = \frac{2.4.6\ldots\ \ 2n}{3.5.7\ldots 2n+1} \qquad [63\,c', e']$$

21. If $m$, $n$ be positive integers and $p=\pi/a$ then $\int_o^a \cos mpx.\cos npx.\,dx=o$ or $\frac{1}{2}a$, according as $m$, $n$ are unequal or equal.

22. If $fx=\frac{1}{2}A_0+A_1\cos px+A_2\cos 2px+A_3\cos 3px+\ldots$ for all values of $x$ from $o$ to $a=\pi/p$, then show that

$$A_n=\frac{2}{a}\int_o^a fx\cos npx.\,dx.$$

[This is Fourier's theorem for expansion of any function in a series of cosines.]

# CHAPTER V

## Successive Differentiation

**125.** Any differential quantity may be itself differentiated, and the result again differentiated, and so on, since any differential is a new variable depending upon the previous independent variables and their differentials. Thus each differentiation introduces as many new independent variables as there are original or *primary* independent variables.

*E. g.*, differentiating (according to Prin. 3) first partially as to $x$ and then partially as to $y$ and adding, we find

$$
\begin{aligned}
d\ .\ xy &= d\ x\ .\ y + x\,dy \\
d^2\ .\ xy &= d^2\,x\ .\ y + dx\ .\ dy \\
&\qquad\qquad + dx\ .\ dy + xd^2y \\
&= d^2\,x\ .\ y + 2\,dx\ .\ dy + x\ .\ d^2y \\
d^3\ .\ xy &= d^3\,x\ .\ y + 2\,d^2x\ .\ dy + dx\ .\ d^2y \\
&\qquad\qquad + d^2x\ .\ dy + 2\,dx\ .\ d^2y + xd^3y \\
&= d^3\,x\ .\ y + 3\,d^2x\ .\ dy + 3\,dx\ .\ d^2y + xd^3y
\end{aligned}
$$

The identity of these successive operations, so far as numerical co-efficients are concerned, with successive multiplications of $a+b$ by $a+b$, shows that we must have always binomial co-efficients, or that

(a) $d^n\,.xy = d^n\,x.y + nd^{n-1}x.dy + n\,(n-1)\,d^{n-2}x\,.\,d^2y + \ldots + xd^n y$. This analogy results from $d^n = (d_x + d_y)^n$, and the fact (Prin. 4 following.) that $d_x$, $d_y$ obey the same *formal* laws of combination as numbers.

**126.** The successive differentiation $d^n = (\lim N\Delta)^n$ can be defined as a single process $\lim N^n \Delta^n$. To explain $\Delta^n$, let $x$ take any variation of $n$ changes from its original value, say $x, x_1, x_2, \dots x_n$. This determines a variation of $n-1$ changes in $\Delta x$, a variation of $n-2$ changes in $\Delta^2 x$, and so on to one value of $\Delta^n x$. If $E$ symbolize the process of changing from one value of a variation to the next (enlarging the variable), so that $Ex = x_1 \quad E^2 x = Ex_1 = x_2, \quad E^3 x = x_3, \ \dots \ E^n x = x_n$, then we shall always have $\Delta = E - 1$, (understanding that a sum of processes, means the summing of results of each process) *i. e.*,

$$\Delta x = (E-1)\,x = Ex - 1\,.\,x = x_1 - x$$
$$\Delta^2 x = (E-1)\,(x_1 - x) = x_2 - x_1 - x_1 + x = x_2 - 2x_1 + x$$
$$\Delta^3 x = (x_3 - 2x_2 + x_1) - (x_2 + 2x_2 - x) = x_3 - 3x_2 - 3x_1 - x$$

and

$$\Delta^n x = (E-1)^n x = x_n - nx_{n-1} + n\frac{(n-1)}{2}x_{n-2} - \dots \mp nx_1 \pm x$$

Note.—It is because $\Delta$, $E$, 1 are processes that are distributive over sums and commutative with each other, that their laws of combination are like those of ordinary numbers. It is different with processes "log" and "$\surd$" that do not obey these laws. *E. g.*, $(\log + \surd)^2 x = (\log + \surd)(\log x + \surd x) = \log(\log x + \surd x) + \surd(\log x + \surd x)$, but not $= \log\log x + 2\log\surd x + \surd\surd x$, since it is not true that $\log(x+y) = \log x + \log y$, $\surd(x+y) = \surd x + \surd y$ and $\log\surd x = \surd\log x$.

Conversely, the $n$ values $\Delta x, \Delta^2 x, \dots \Delta^n x$ determine the variation of $x$, *viz.*,

$$x_n = E^n x = (1+\Delta)^n x = x + n\Delta x + \frac{n(n-1)}{2}\Delta^2 x + \ldots$$
$$+ n\Delta^{n-1}x + \Delta^n x.$$

**127.** In the case of an independent variable $x$, the $n$ successive differences $\Delta x$, $\Delta^2 x$, ... $\Delta^n x$ are assignable at will, and can each be made to approach zero in such a way that for any proportional factor $N$ that approaches infinity, the proportional differences $N\Delta x$, $N^2\Delta^2 x$, ... $N^n\Delta^n x$ shall approach any assigned limits $dx$, $d^2x$, ... $d^n x$. At the same time, the successive proportional differences $N\Delta w$, $N^2\Delta^2 w$ ... $N^n\Delta^n w$, of a *successively differentiable* dependent variable $w$, must approach limits $dw$, $d^2w$, ... $d^n w$, that depend only upon the values of the independent variables and their successive differentials.

*E.g.*, $\Delta^2 xy = x_2 y_2 - 2x_1 y_1 + xy$

$= (x+2\Delta x+\Delta^2 x)(y+2\Delta y+\Delta^2 y) - 2(x+\Delta x)(y+\Delta y)+xy$

$= x\Delta^2 y + 2\Delta x\Delta y + y\Delta^2 x + 2\Delta x\Delta^2 y + 2\Delta^2 x\Delta y + \Delta^2 x\Delta^2 y$

so that $d^2 . xy = \lim N^2\Delta^2 .xy = xd^2y + 2\,dx\,dy + yd^2x.$

### Partial Differentiation

**128.** Differentiation under the suppositions that certain variable quantities are constants is called *partial differentiation.* When the suppositions affect only independent variables and not all of those, then partial differentiation of equals give equals. Thus in $(x+y)^2 = x^2 + 2xy + y^2$, we may consider either $x$ or $y$ to vary alone, or both to vary together so that $xy$ is constant, and the differentials of each member are equal under any of these suppositions. But in

$x^2 + y^2 = 4$, where both variables must change in order to maintain the equality, partial differentiation as to $x$ or $y$ does not give an equation that follows from the given one.

Prin. 4. *Two successive partial differentiations of any function are commutative in order of operation.*

In symbols $d_1\, d_2\, w = d_2\, d_1\, w$, where $d_1'$ affects certain variables $x$, ... of $w$, and $d_2$ affects certain variables $y$, ... of $w$.

By Prin. 3 $\quad d_2 w = (d_y + \dots)\, w = d_y\, w + \dots$
and $\quad d_1\, d_2 w = (d_x + \dots)\, d_2\, w = d_x\, d_y\, w + \dots$
Similarly $\quad d_1 w = (d_x + \dots)\, w = d_x\, w + \dots$
and $\quad d_2\, d_1 w = (d_y + \dots)\, d_1 w = d_y\, d_x\, w + \dots$

It therefore only remains to prove the principle for partial differentiations as to any two variables $x$, $y$, and this is done at once by

$$\begin{aligned} d_x\, d_y f(x, y) &= d_x \lim N[f(x, y') - f(x, y)] && \text{(definition)} \\ &= \lim N\,[d_x f(x, y') - d_x f(x, y)] && \text{(Prin 2)} \\ &= d_y\, d_x\, f(x, y) && \text{(definition)} \end{aligned}$$

By dividing this result by $dx\, dy$, we find,

(a) $$\frac{\partial}{\partial x}\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}\frac{\partial w}{\partial x}$$

**129.** It is not possible to mark a differential symbol so as to show all possible suppositions under which the differentiation is taken, and *form* is often used instead of marks for the purpose. It follows that changes of form that are legitimate when the differentials are sufficiently marked to indicate their significance, are not legitimate when the form is changed so as to lose its assigned significance as a mark of a certain kind of differentiation. For example, the early practice was, with a function $w$ of $x, y$, to use $dw/dx$, $dw/dy$ as forms for *partial* differentiation as to $x, y$, so that in this notation $dw = \frac{dw}{dx}dx + \frac{dw}{dy}dy$. To cancel $dx$, $dy$ here gives the incorrect result $dw = 2dw$; but if we mark the differentiations by subscripts, then $dx$, $dy$, may be cancelled, giving $dw = d_x w + d_y w$, a correct result.

**130.** Again, if $y$ be a function of $x$, whose successive derivatives are $y', y'', \ldots$ so that $dy = y'dx$, $dy' = y''dx, \ldots$ then $d^2y = y''\,dx^2 + y'\,d^2x$,

$$d^3y = y'''\,dx^3 + 3y''\,dx\,d^2x + y'\,d^3x, \text{ etc.}$$

If we suppose $dx$ to be constant in successive differentiations then $d^2y = y''\,dx^2$, $d^3y = y'''\,dx^3, \ldots$ and the latter forms are understood as *indicating partial differentiation with dx assumed constant;* so that the quotients $d^2y/dx^2 = y''$, $d^3y/dx^3 = y'''$ become *abbreviated* forms for the successive derivatives of $y$ as to $x$. The full forms for these derivitives are

$$\frac{d}{dx}\frac{dy}{dx}, \frac{d}{dx}\frac{d}{dx}\frac{dy}{dx},$$

in which $d$ is unrestricted. Since every differentiation, in these successive derivations, is performed upon a function of $x$ alone, it follows that it is independent of differentials and may be performed under the supposition that any differential we please is constant; in particular $dx =$ constant reduces them by Prin. 2 to the above abbreviated forms. If $x$ is not the independent variable but is a function of the independent variable $z$, say $x = fz$, then we may transform derivatives as to $x$ into derivatives as to $z$ by substituting $dx = f'z\,dz$ in the *complete* forms, but not in the *abbreviated* ones. Thus, $\frac{d}{dx}\frac{dy}{dx} = \frac{d}{f'zdz}\,\frac{d}{f'zdz}$; but $d^2y = y''\,dx^2$ does not become $d^2y = y''\,f'z^2\,dz^2$ because the accepted significance of the latter form is that $dz =$ constant whereas for the first, it is that $dx =$ constant, so that the two symbols $d^2y$ are not symbols of the same quantity. The true equality resulting from $dx = f'z\,dz$ is here, $(d^2y)_{dx\ \mathrm{constant}} = y''\,f'z^2\,dz^2$.

We have always $\frac{d^2y}{dx^2} = \frac{d}{dx}\cdot\frac{dy}{dx} = \frac{d^2y\,dx - d^2x\,dy}{dx^3}$ in which, as pointed out above, the differentiations of the last form are unrestricted, and may be made as to any differential a constant. In particular to make $y$ the independent variable, we let $dy =$ constant, and so find

$$\frac{d^2y}{dx^2} = -\frac{dy^3}{dx^3}\cdot\frac{d^2x}{dy^2}.$$

## Examples IV

1. $y = x^2 e^x$; $\quad d^2y/dx^2 = e^x(x^2 + 4x + 4)$
2. $y = x^5$; $\quad d^5y/dx^5 = 5!$
3. $y = x^3 \log x$; $\quad d^4y/dx^4 = 5/x$
4. $y = \log \sin x$; $\quad d^2y/dx^2 = -\csc x^2$
5. $y = e^{-mx}(a \cos nx + b \sin nx)$;

$$\frac{d^2y}{dx^2} + 2m\frac{dy}{dx} + (m^2 + n^2)\,y = o$$

6. $y = e^{-mx}(a + bx)$;

$$\frac{d^2y}{dx^2} + 2m\frac{dy}{dx} + m^2y = o$$

7. $y = a e^{-mx} + b e^{-nx}$;

$$\frac{d^2y}{dx^2} + (m+n)\frac{dy}{dx} + mny = o.$$

8. Solve the equation $\dfrac{d^2y}{dx^2} + A\dfrac{dy}{dx} + By = o$ where $A$, $B$ are constants.

Try $y = e^{cx}$, whence $c^2 + Ac + B = o$ to determine $c$. Show that the sum of two solutions, each multiplied by an arbitrary constant, is a solution, and thence, if $c = -m, -n$, derive the solution of Ex. 7. This solution is general, because it involves two arbitrary constants $a$, $b$, such as would be obtained by two successive anti-differentiations. If $m = n$ show that besides $e^{-mx}$ also $xe^{-mx}$ is a solution, so that the general solution is that of Ex. 5. If $c = -m + n\sqrt{-1}$, $-m - n\sqrt{-1}$, then replace the two solutions $e^{cx}$ by their sum multiplied by $1/2$ and their difference multiplied by $1/2\sqrt{-1}$, and so obtain the general solution of Ex. 5.

9. If $y = f_1x$ be a solution of $\dfrac{d^2y}{dx^2} + A\dfrac{dy}{dx} + By = fx$ find the general solution.

[Let $y = f_1x + z$ be the general solution; whence $z$ is found from $\dfrac{d^2z}{dx^2} + A\dfrac{dz}{dx} + Bz = o$.]

10. Solve $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 26y = 154\cos 4x + 8\sin 4x$.

Ans. $y = 9\cos 4x + 8\sin 4x + e^{-x}(a\cos 5x + b\sin 5x)$.

11. Show that $\frac{d^n}{dx^n} u v = \frac{d^n u}{dx^n} v + n \frac{d^{n-1} u}{d x^{n-1}} \frac{dv}{dx} + \dots$

$$= \left\{\frac{d_u}{dx} + \frac{d_v}{dx}\right\}^n . uv.$$

12. Show that $\frac{d^n}{dx^n} e^{ax} f x = e^{ax} (a + \frac{d}{dx})^n f x.$

[Verify for $n = 1$ and thence for $n = 2, 3, \dots$]

13. $\frac{d^n}{dx^n} e^{ax} x^2 = e^{ax} [a^n x^2 + 2n a^{n-1} x + n (n-1) a^{n-2}]$

14. Verify the following transformations of independent variable, and solve the equations.

$$\frac{d^2 y}{dx^2} - 3 \frac{dy^2}{dx^2} - 2x \frac{dy^3}{dx^3} = o \text{ to } \frac{d^2 x}{dy^2} + 3 \frac{dx}{dy} + 2x = o$$

$$\frac{d^2 y}{dx^2} + \frac{2x}{1 + x^2} \frac{dy}{dx} + \frac{y}{(1 + x^2)^2} = o \text{ to } \frac{d^2 y}{dz^2} + y = o, x = \tan z.$$

**Successive Integration**

**131.** We consider certain "multiple" integrals in two or more variables, where the integration is partial with respect to each variable in turn, and as if the remaining variables were constants, while the limits of variation of each variable are constants, or at most functions of the variables that are assumed as constants. The integral is written so that the order of successive integration is from right to left, so that each integration reduces the multiple integral to one of next lower multiplicity. Thus

$$\int_a^u \int_b^v f(x, y)\, dx\, dy = \int_a^u \left\{ \int_b^v f(x, y)\, dy \right\} dx.$$

is a double integral. The bracketed integral is first evaluated, with $x$ as a constant, and $b$, $v$ are at most functions of $x$, so that the bracketed integral is simply a $\phi x$ term in a single integral. Similarly

$$\int_a^u \int_b^v \int_c^w F(x, y, z)\, dx\, dy\, dz$$
$$=\int_a^u \int_b^v \left\{ \int_c^w F(x, y, z)\, dz \right\} dx\, dy$$

is a triple integral. The bracketed integral is first evaluated, with $x$, $y$ as constants, and $c$, $w$ are at most functions of $x$, $y$, so that the bracketed integral is simply an $f(x, y)$ term in a double integral. Similarly for multiple integrals in four or more variables.

**132.** The single, double, and triple integrals have geometric representation as the limits of sums of differences over a line, surface, and volume respectively. We illustrate by examples worked out in detail, showing also in the first illustration the sums considered, of which the integrals are limits for continuous variation. The student should make the drawings as described.

**133.** *Find the moment of inertia of a rectangular parallelopiped of edges $a$, $b$, $c$, about the edge $c$ as axis.*

Let $OA=a$, $OB=b$, be the two edges in the plane of the diagram. Confine attention to the rectangle $AB$, knowing that a length $c$ of the volume is above every point $P$. Take $OA$, $OB$ as axes of $x$, $y$. Lay off $OL=x$, $OL'=x'$, $LP=y$, $L'P'=y'$, where $P$, $P'$ are neighboring points within the rectangle $AB$. Then $c\Delta x\, \Delta y$ is the difference volume whose base is the rectangle $PP'$ and $(x_1^2 + y_1^2) c\Delta r \Delta y$ is its moment of inertia as to the axis $OC$, where $(x_1, y_1)$ is some point of the rectangle $PP'$. Here, $x_1^2 + y_1^2$ is a function, $\phi(x, \Delta x, y, \Delta y)$, that reduces to $x^2 + y^2$ when $\Delta x = o$, $\Delta y = o$. If we assume $x$, $\Delta x$ to be constants and give $y$ any variation from $o$ to $b$, then the sum of the difference volumes $PP'$, is $\Sigma_o^b c\Delta x\, \Delta y, = bc\Delta x$, the difference volume whose base is the rectangle $LL'\,Q'\,Q$ of altitude $b$ and base $\Delta x$; and the sum of the moments of inertia of the difference volumes $PP'$ is $\Sigma_o^b (x_1^2 + y_1^2)\, c\Delta x \Delta y =$ the moment of inertia of the difference volume $LQ'$. Next give $x$ any variation from $o$ to $a$, and then the sum of the difference volume $LQ'$, is $\Sigma_o^a bc\, \Delta x = a\,b\,c =$ the entire volume; and the sum of their moments of inertia is the entire moment of inertia, $\Sigma_o^a \Sigma_o^b (x_1^2 + y_1^2) c\Delta x \Delta y$. The results hold for any variation, first of $y$ from $o$ to $b$ with $x$, $\Delta x$ constant, and then of $x$ from $o$ to $a$; and in particular for continuous variations, in which the sums become integrals of the differentials corresponding to the vanishing differences. Thus $c\,dx\,dy$ is the differential volume $P$, and $(x^2 + y^2)\, c\, dx\, dy$ is its moment of inertia. Also, $\int_o^b c\, dx\, dy = bc\, dx$ is the differential valume $LQ$, while $\int_o^b (x^2 + y^2)\, c\, dx\, dy = bc\ (x^2 + \frac{b^2}{3})\, dx$ is its moment of inertia; and finally, $\int_o^a \int_o^b c\, dx\, dy = \int_o^a bc\, dx = abc$,

is the total volume, and $\int_o^a \int_o^b (x^2+y^2)c\,dx\,dy =$
$= \int_o^a bc\,(x^2 + \frac{b^2}{3})\,dx = a\,b\,c\frac{a^2+b^2}{3}$ is the total moment of inertia.

**134.** *Find the moment of inertia of a right cylinder about its axis, $OC = c$, where the base is an elliptic quadrant of radii $OA = a$, $OB = b$.*

As before $c\,dx\,dy$, $c\,(x^2+y^2)\,dx\,dy$ are the differential volume at $P$ and its moment of inertia; and these integrals from $y = o$ to $y = y_1 = LK = b\sqrt{(a^2-x^2)}/a$ = the ordinate of the point $K$ on the ellipse $AB$ whose abscissa is $x = OL$, are

$$\int_o^{y_1} c\,dx\,dy = c\,y_1\,dx,$$

$$\int_o^{y_1} c\,(x^2+y^2)\,dx\,dy = c\,(x^2\,y_1 + \frac{y_1^3}{3})\,dx.$$

These are therefore respectively the differential volume $LK$ and its moment of inertia. Finally

$$\text{volume} = \int_o^a c\,y_1\,dx = \frac{\pi\,abc}{4} \qquad \text{[Ex. 14, p. 43.]}$$

$$\text{mom. iner.} = \int_o^a c\,(x^2\,y_1 + \frac{x_1^3}{3})\;dx$$

$$= \frac{\pi\,a\,b\,c}{4}\,\frac{a^2+b^2}{4} \qquad \text{[Ex. 16, p. 43.]}$$

**135.** *Find the moment of inertia of the preceding elliptic cylinder about the axis $OA$.*

Show the axis $OC$ and the cylinder $COAB$ in perspective on the plane of the diagram. Draw a plane through the ordinate $LK$ perpendicular to $OA$, and therefore intersecting the cylinder in elements $KN$, $LM$; draw $RQ$ parallel to $LM$ and intersecting $LK$ in $R$, $MN$ in $Q$; and on $RQ$ take any point $P = x, y, z = (OL, LR, RP)$. The differential volume $P$ is $dx\, dy\, dz$, and its moment of inertia as to $OA$ is $(y^2 + z^2)\, dx\, dy\, dz$. Integrating from $z = o$ to $z = c$ we find the differential volume $RQ$ and its moment of inertia,

$$\int_o^c dx\, dy\, dz = c\, dx\, dy,$$

$$\int_o^c (y^2 + z^2)\, dx\, dy\, dz = (y^2 c + \frac{c^3}{3})\, dx\, dy.$$

Integrating again from

$$y = o \text{ to } y = y_1 = LK = b\sqrt{a^2 - x^2} / a$$

we find the differential volume $KLMN$ and its moment of inertia,

$$c y_1\, dx, \quad c y_1 \frac{y_1^2 + c^2}{3} dx$$

These are results that can be found directly from geometry and Art. 134. Integrating then from $x = o$ to $x = a$, we find the total volume and its moment of inertia,

$$\tfrac{1}{4}\pi abc, \quad \tfrac{1}{4}\pi abc\,(b^2 / 4 + c^2 / 3.)$$

**136.** *Find the volume and moment of inertia about $OC$ of the octant of an ellipsoid of radii $OA = a$, $OB = b$, $OC = c$.*

The figure is shown in plane diagram by projections of quadrants of the ellipses $AB$, $BC$, $CA$; the section by a plane perpendicular to $OA$ through the ordinate $LK$ of arc $AB$ is an elliptic quadrant $KM$ from $AB$ to $CA$;

draw in this plane $RQ$ parallel to $LM$ from $LK$ to arc $KM$, and we have since $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ for any point $(x, y, z)$ on the ellipsoid, $LK=b\sqrt{(a^2-x^2)}/a$, $RQ=c\sqrt{(1-\frac{x^2}{a^2}-\frac{z^2}{b^2})}$. Take on $RQ$ any point $P=(x, y, z)=(OL, LR, RP)$. The volumes and moments of inertia about $OC$ are,

$$\int_o^a\int_o^{LK}\int_o^{RQ} dx\,dy\,dz$$

and the same integration of $(x^2+y^2)\,dx\,dy\,dz$.

The results of the first two partial integrations may be found geometrically from Art. 134, applied to the elliptic cylinder of length $dx$ on the base $LKM$, and are

$\frac{\pi}{4}LK\,.\,LM\,dx$, $\frac{\pi}{4}LK\,.\,LM\,dx.\,\frac{LK^2+LM^2}{4}$ or [since $LM=c\sqrt{(a^2-x^2)}/a$], $\frac{\pi bc}{4a^2}(a^2-x^2)\,dx$, $\frac{\pi bc\,(b^2+c^2)}{16\,a^2}(a^2-x^2)^2\,dx$, and the integrals of these from $x=o$ to $x=a$ give finally $\frac{\pi abc}{6}$, $\frac{\pi abc}{6}\frac{(a^2+b^2)}{5}$.

**137.** If $m$ denote the volume (or mass of a homogeneous volume) then the rectangular parallelopiped, elliptic cylinder, and ellipsoid, whose semi-axes of symmetry are $OA=a$, $OB=b$, $OC=c$, and of which the volumes of the preceding articles are octants, have the following moments of inertia

Parallelopiped: $m\dfrac{a^2+b^2}{3}$ about $OC$; etc.

Cylinder: $m\dfrac{a^2+b^2}{4}$ about $OC$;

$m\left(\dfrac{b^2}{4}+\dfrac{c^2}{3}\right)$ about $OA$, etc.

Ellipsoid: $m\dfrac{a^2+b^2}{5}$ about $OC$; etc.

These are easily remembered, and are useful, especially in connection with the theorem of Art. 139.

**138.** *Center of gravity.* The differential element of mass at $(x, y, z)$ called a *particle P*, is $u\,dx\,dy\,dz$ where $u$ is in general a function of $x, y, z$ denoting the *density* at that point. If $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ be the center of mass (center of gravity) then computing moments as to the planes $yz$, $zx$ $xy$ directly, and by the sum of the moments of the particles, we find

$$m\,\mathbf{x}=\int\int\int ux\,dx\,dy\,dz$$

$$m\,\mathbf{y}=\int\int\int uy\,dx\,dy\,dz$$

$$m\,\mathbf{z}=\int\int\int uz\,dx\,dy\,dz$$

**139.** *The moment of inertia of a given mass about a given axis is equal to the moment of inertia of the mass about a paralled axes through the center of mass plus the product of the mass and the square of the distance between the axes.*

Let $OZ$ be the axis through the center of mass, and let $OA=a$, be the distance between the axcs. We have $\int\int\int ux\,dx\,dy\,dz=m\,\mathbf{x}=o$ by hypothesis; and therefore $m\,k^2 = \int\int\int [(x-a)^2 + y^2]\,u\,dx\,dy\,dz =$ $\int\int\int (x^2+y^2)\,u\,dx\,dy\,dz + a^2 \int\int\int u\,dx\,dy\,dz =$ $m\,k_0^2 + m\,a^2$.

**140.** A cylinder with elements perpendicular to the $x, y$ plane intersects a surface $z = f(x, y)$; required the surface area intercepted by the cylinder.

If $\Delta^2 S$ be the portion of the surface intercepted by the cylinder on the rectangle $\Delta x\,\Delta y$ as base, the corresponding differential element of surface $d^2 S$ is intercepted on the tangent plane to the surface at $P=(x, y, z)$ by the cylinder on $dx\,dy$ as base; so that if $\gamma$ be the angle between this tangent plane and the plane $x\,y$, *i. e.*, the angle between the axis of $z$ and the normal to the surface at $P$, we have $d^2 S \cos\gamma = dx\,dy$ so that $d^2 S = dx\,dy \sec\gamma$, and $S = \int\int \sec\gamma\,dx\,dy$, over the base of the given cylinder on the $x\,y$ plane. To find $\sec\gamma$ in terms of $x, y$ observe that if $w = o$ be the equation of a surface, then for variations of $P$ on that surface $w = o$ or $\frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz = o$. This shows that the line $PN$ whose components on the axes are $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}$ is perpendicular to the tangent line $PS = ds$, whose components are $dx, dy, dz$.* .

Thus $PN$ is a normal to the surface at $P$ since it

*If $r, r'$ be two lines that make an angle $A$ with each other and angles $a, b, c,\ a', b', c'$, with the axes, then eqating the projection of $r$ upon $r'$ to the sum of the projections of the components of $r$ upon $r'$, we have

$$r \cos A = r \cos a . \cos a' + r \cos b . \cos b' + r \cos c . \cos c'$$

or $r r' \cos A = l\,l' + m\,m' + n\,n'$ in terms of the components of $r, r'$; if this is zero then $A$ is a right angle.

is perpendicular to every tangent line $PS$, and $PN \cos\gamma = \partial w/\partial z, = 1$ when $w = z - f(x, y)$. Also $PN^2 = \left.\frac{\partial w}{\partial x}\right|^2 + \left.\frac{\partial w}{\partial y}\right|^2 + \left.\frac{\partial w}{\partial z}\right|^2, = 1 + \left.\frac{\partial z}{\partial x}\right|^2 + \left.\frac{\partial z}{\partial y}\right|^2$ where $z = f(x, y)$.

EXAMPLES.

1. Find the center of gravity of an arc of a circle of radius $a$ and length $l$.

[Take the center as origin and axis of $y$ parallel to the chord, whose length is $k = 2a\sin^{-1}(l/2a)$. Then

$$\mathbf{x} = \int_o^l x\,ds \Big| \int_o^l ds = \int_{-\frac{1}{2}k}^{\frac{1}{2}k} a\,dy \Big| l = a\frac{k}{l}.$$

since $y$ varies from $-\frac{1}{2}k$ to $\frac{1}{2}k$ when $s$ varies from $o$ to $l$.

$$\mathbf{y} = \int_o^l y\,ds \Big| \int_o^l ds = \int_h^h -a\,dx \Big| l = 0.$$

2. Find the center of gravity of a straight wire whose density varies uniformly from end to end.

Let $a, b$, be the densities at the ends, and $l$ be the length of the wire; then the density at distance $x$ from the first end is $u = a + (b-a)x/l$; and

$$\mathbf{x} = \int_o^l ux\,dx \Big| \int_o^l u\,dx = \tfrac{1}{3}l\,(a+2b)/(a+b)$$

3. Find the center of gravity of the first quadrant arc of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

$$\mathbf{x} = \mathbf{y} = \tfrac{2}{5}a.$$

4. Find the center of gravity of the first quadrant area of the circle $x^2 + y^2 = a^2$.

$$\mathbf{x}=\int_{o}^{a}\int_{o}^{\sqrt{(a^2-x^2)}}x\,dx\,dy\Big|\int_{o}^{a}\int_{o}^{\sqrt{(a^2-x^2)}}dx\,dy$$

$$=\frac{4a}{3\pi}=\mathbf{y}.$$

5. Find the center of gravity of the first quadrant area of the hypocycloid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

6. Find the center of gravity of the area of the cardioid $r=2a(1-\cos\theta)$.

7. Find thc center of gravity of the parabloic area $y^2=4ax$ from $x=o$.

8. Find the center of gravity of a hemisphere whose density varies as the distance from the center; find also its radius of gyration about its axis of symmetry.

9. Find the center of gravity and the radius of gyration of the volume generated by the revolution of $y^2=4ax$ about the axis of $x$ (from $x=o$).

10. Find the radius of gyration of a sphere about a tangent line as axis. (Arts. 137, 239).

11. Compare and verify when necessary the following formulas for straight and rotary motion of a rigid body; $dm$ = mass of a differential particle; $v$ = its linear velocity; $r$ = its distance from the axis in rotary motion. The integration extends to every particle of the body, and for this integration $v$ = constant in linear motion, and $v/r=w=$ constant in rotary motion; $v$ and $w$ are, however, variable with the time.

| STRAIGHT MOTION | ROTARY MOTION |
|---|---|
| $m=$inertia (mass) | I$=$moment of inertia |
| $=\int dm$ | $=\int r^2\,dm$ |
| $v=$velocity | $w=v/r=$angular vel. |
| $mv=$momentum | I$w=$mo. of momentum |
| $=\int v\,dm$ | $=\int rv\,dm$ |
| $\dfrac{dv}{dt}=$acceleration | $\dfrac{dw}{dt}=$angular acceleration |
| $m\dfrac{dv}{dt}=$force | $\mathrm{I}\dfrac{dw}{dt}=$moment of forces |
| $=\int\dfrac{dv}{dt}\,dm$ | $=\int r\dfrac{dv}{dt}\,dm$ |
| $\frac{1}{2}mv^2=$kinetic energy | $\frac{1}{2}\mathrm{I}w^2=$kinetic energy |
| $=\int\frac{1}{2}v^2\,dm$ | $=\int\frac{1}{2}v^2\,dm$ |

In finding the moment of the forces in rotary motion the tangential acceleration of $dm$, viz, $\dfrac{dv}{dt}$, is the only effective component, since the normal component $v^2/r$ passes through the axis. (See Art. 86). In straight motion there is no normal component of acceleration since the curvature of a straight line is zero (or its radius of curvature infinite).

12. Find the surfaces cut from one another by a right circular cylinder of radius $a$ and a sphere of radius $2a$, whose center is on the cylindrical surface.

13. Find the volume enclosed in Ex. 12.

14. Find the volume enclosed by the surface

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}=a^{\frac{2}{3}}$$

15. Show that if an arc of length $s$ be revolved about an axis in its plane, the surface described is the product of $s$ into the length of path of the center of gravity of $s$. State and prove a similar theorem for revolution of a plane area about an axis in its plane.

16. Find the volume and surface of the anchor ring, generated by revolving a circle of radius $b$, about an axis in its plane at distance $a$ from the center.

17. In polar co-ordinates $(r, \theta, \phi)$ of a point $P$ with reference to an initial line $OA$ and initial plane $OAB$, we have

$$r=OP,\ \theta=\angle AOP,\ \phi=$$

diedral angle between $OAP$ and $OAB$. Show that the differential element of volume is

$$\begin{aligned} dV &= dr\,.\,rd\theta\,.\,r\sin\theta\, d\phi. \\ &= r^2 \sin\theta\, dr\, d\theta\, d\phi. \end{aligned}$$

## Rules of Differentiation

I. $d.u^l = lu^{l-1}du.$

(a) $d.u^l v^m = u^{l-1}v^{m-1}(lv\,du + mu\,dv).$

(b) $d.u^l / v^m = u^{l-1}(lv\,du - mu\,dv)/v^{m+1}.$

II. $d.a^y = a^y \log a\,dy;$ $\qquad d.e^y = e^y\,dy.$

(a) $d.u^y = yu^{y-1}du + u^y \log u.\,dy.$

III. $d\log_a v = dv/v\log a;$ $\qquad d\log v = dv/v.$

(a) $d\log(x+\sqrt{x^2+ca^2}) = dx/\sqrt{(x^2+ca^2)}.$

(b) $d\log\dfrac{x}{a+\sqrt{a^2+cx^2}} = \dfrac{a\,dx}{x\sqrt{(a^2+cx^2)}}.$

(c) $d\log\dfrac{a+x}{a-x} = \dfrac{2a\,dx}{a^2-x^2}.$

IV $d\sin v = \cos v.\,dv;$ $\qquad d\cos v = -\sin v.\,dv.$

$d\tan v = \sec v^2\,dv;$ $\qquad d\cot v = -\csc v^2\,dv.$

$d\sec v = \sec v.\tan v.\,dv;$ $\qquad d\csc v = -\csc v.\cot v.\,dv.$

V $d\sin^{-1}\dfrac{x}{a} = \dfrac{dx}{\sqrt{(a^2-x^2)}} = -d\cos^{-1}\dfrac{x}{a}.$

$d\tan^{-1}\dfrac{x}{a} = \dfrac{a\,dx}{a^2+x^2} = -d\cot^{-1}\dfrac{x}{a}.$

$d\sec^{-1}\dfrac{x}{a} = \dfrac{a\,dx}{x\sqrt{(x^2-a^2)}} = -d\csc^{-1}\dfrac{x}{a}.$

$d\,\mathrm{vers}^{-1}\dfrac{x}{a} = \dfrac{dx}{\sqrt{(2ax-x^2)}} = d\sin^{-1}\dfrac{x-a}{a}$

Prin 3. $df(x,y) = d_x f(x,y) + d_y f(x,y).$

**Rules of Integration**

I $\quad d^{-1}.u^l du = u^{l+1}/(l+1)$.

III $\quad d^{-1}.du/u = \log u$, or $\log - u$, or $\log c u$.

III (a) $\displaystyle\int \frac{dx}{\sqrt{(x^2+ca^2)}} = \log(x + \sqrt{x^2+ca^2})$, etc.

V $\quad \displaystyle\int \frac{dx}{\sqrt{(a^2-x^2)}} = \sin^{-1}\frac{x}{a}$.

V $\quad \displaystyle\int \frac{dx}{\sqrt{(2ax-x^2)}} = \text{vers}^{-1}\frac{x}{a}$, or $\sin^{-1}\dfrac{x-a}{a}$.

V $\quad \displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$.

III (c) $\displaystyle\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\frac{a+x}{a-x}$.

III (b) $\displaystyle\int \frac{dx}{x\sqrt{(a^2+cx^2)}} = \frac{1}{a}\log\frac{x}{a+\sqrt{x^2+cx^2}}$.

V $\quad \displaystyle\int \frac{dx}{x\sqrt{(x^2-a^2)}} = \frac{1}{x}\sec^{-1}\frac{x}{a}$.

II $\quad d^{-1}.\, a^x\, dx = a^x/\log a$; $\quad d^{-1}.\, e^x\, dx = e^x$.

IV $\quad d^{-1}.\sin v\, dv = -\cos v$; $\quad d^{-1}.\cos v\, dv = \sin v$.

III $\quad d^{-1}.\tan v\, .\, dv = -\log\cos v$;

$\quad d^{-1}.\cot v\, dv = \log\sin v$.

III $\quad d^{-1}.\sec v\, dv = \log(\sec v + \tan v)$;

$\quad d^{-1}.\csc v\, dv = \log(\csc v - \cot v)$.

IV $\quad d^{-1}\sec v^2\, dv = \tan v$; $\quad d^{-1}\csc v^2\, dv = -\cot v$.

IV $\quad d^{-1}\sec v\tan v\, dv = \sec v$;

$\quad d^{-1}\csc v\cot v\, dv = -\csc v$.

$\quad d^{-1}.\sin v^2\, dv = \frac{1}{4}(2v - \sin 2v)$;

$\quad d^{-1}\cos v^2\, dv = \frac{1}{4}(2v + \sin 2v)$.

Inv. Prin. 3. $\quad d^{-1}.\, d_x f(x,y) = f(x,y) - d^{-1} d_y f(x,y)$.